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Characterization of extra polyloop-I

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Introduction

In this study, we present new non-associative algebraic hyperstructures namely, multiloop, polyquasigroup, and polyloop. We extend this notion of hyper-algebraic structures to coin a special class of polyloop called extra polyloop. The first extra identity in classical loop theory was adopted to conceptualize extra polyloop-1, which is characterized by any of seven identities, which were found to be equivalent.
The flexibility law and some basic algebraic properties satisfied by each of these equivalent identities were established. Another notion like autotopism of extra polyloop-1 was also established. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. For any non-commutative (groupoid, quasigroup, loop), commutative and non-commutative (polygroupoid, polyquasigroup, polyloop) can be constructed.

Preliminaries and Background of Study

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extra loop [F. Fenyves [8, 9]]

A loop $(G,\cdot)$ is said to be an extra loop if and only if it satisfies the identity $(xy\cdot z)x = x(y\cdot zx)$ for all $x,y,z\in G$. Identities (i) - (iii) were shown to be equivalent:

(i) $(xy\cdot z)x = x(y\cdot zx)$

(ii) $yz\cdot yx = y(zy\cdot x)$

(iii) $(x\cdot yz)y = xy\cdot zy)$ $\forall$x,y,z $\in G$

Consequently, in the study of non-associative algebraic hyperstructures, we present analogous results which characterize the extra polyloop-1 identity with seven other equivalent identities.

Hypergroupoid, Hypergroup

In 1934, during the 8th Congress of Scandinavian Mathematicians, F. Marty [10] characterized hypergroups as a natural generalization of the idea of a group. The study of hyperstructure was further exhibited by P. Corsini [4]. In [12], polyquasigroups and polyloops were introduced, and extensively studied and some of their algebraic properties were established.

Definition

Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation (multivalued operation). The couple $(H, \circ)$ is known as a hypergroupoid.

Let G be a group and H be any subgroup of G. Then, $G/H = \{xH \mid x \in G\}$ becomes a hypergroup where the hyperoperation is defined in a usual manner:

$$aH \circ bH = \{cH \mid c \in aH \cdot bH\} \text{ for all } a, b \in G.$$

Definition

An hypergroupoid $(H, \circ)$ is the pair of a non-empty set H with an hyperoperation $\circ : H \times H \to P(H) \setminus \{\emptyset\}$ defined on it. An hypergroupoid $(H, \circ)$ is called a semihypergroup if

(i) it obeys the associativity law $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v$$

An hypergroupoid $(H, \circ)$ is called a quasihypergroup if

(ii) it obeys the reproduction axiom $x \circ H = H = H \circ x$ for all $x \in H$.

An hypergroupoid $(H, \circ)$ is called an hypergroup if it is a semihypergroup and a quasihypergroup.

H_v-group

Definition

A hypergroupoid $(H, \circ)$ is called an H_v-group if it is a quasihypergroup and it obeys the weak associativity (WASS) condition

(iii) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$ for all $x, y, z \in H$.

A hypergroupoid $(H, \circ)$ is called an Marty-Moufang hypergroup (H_m-group) if it is a quasihypergroup and it obeys the Moufang identity

(iv) $(x \circ y) \circ (z \circ x) = x \circ ((y \circ z) \circ x)$ for all $x, y, z \in H$.

We often see the reproduction axiom used in the form: Given a, $b \in H$, there exist $x, y \in H$ such that $b \in a \circ x$ and $b \in y \circ a$. Hence, an hypergroup (of Marty) is equivalent to a multigroup of Dresher and Ore [7]

Definition

A hypergroup is a couple $(H,\circ)$, where $\circ : H\times H \longrightarrow P^{*}(H)$, such that the following conditions hold for all x,y,z of H:

1. $(x\circ y)\circ z = x\circ(y\circ z)$ for all $x,y,z\in H$ which means that

$$\bigcup_{u\in x\circ y} u\circ z = \bigcup_{v\in y\circ z} x\circ v$$

2. $H\circ x = x\circ H = H$, where

$$H\circ x = \bigcup_{h\in H} h\circ x \text{ and } x\circ H = \bigcup_{h\in H} x\circ h$$

This condition is called *the reproduction axiom*.

Polygroup

Definition [Davvaz [6]]

A polygroup is a system $\wp = < P, \cdot, e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

(P1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(P2) $e \cdot x = x \cdot e = x$

(P3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

A polygroup is a special type of hypergroup.

Polyquasigroup, Polyloop and Multiloop

Definitions

Let $\mathscr{P} = (G, \cdot)$ be a polygroupoid such that $/ : G \times G \to P^*(G)$ and $\backslash : G \times G \to P^*(G)$.

(a) If (i) $y \in x \cdot (x \backslash y)$ (ii) $y \in x \backslash (x \cdot y)$ (iii) $y \in (y/x) \cdot x$ (iv) $y \in (y \cdot x)/x)$ then $(G, \cdot, \backslash, /)$ will be called a polyquasigroup.

(b) If $x \cdot e = e \cdot x = x$ for all $x \in G$ and $(G, \cdot, \backslash, /)$ is a polyquasigroup, then $(G, \cdot, \backslash, /, e)$ will be called a polyloop.

(c) $x \in x \cdot e = e \cdot x$ for all $x \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyquasigroup. Then $(P, \cdot, \diagdown, \diagup, e)$ will be called a multiloop.

(d) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyloop. Then $(P, \cdot, \diagdown, \diagup)$ will be called an associative polyloop.

Definition

Let $(G,\cdot,/,\backslash,e)$ be a polyloop, then $(G,\cdot,/,\backslash,e)$ is called an extra polyloop-1 if it satisfies the identity

$$(xy\cdot z)x = x(y\cdot zx)$$

for all $x,y,z\in G$

Main Results

We now present some results on the algebraic properties and characterization of extra polyloop-1.

Theorem

Let $(G, \cdot, /, \backslash, e)$ be a polyloop then, $(G, \cdot, /, \backslash, e)$ is an extra polyloop-1 iff any of the following is true in $(G, \cdot, /, \backslash, e)$ for all $x, y, z \in G$; $X, Y, Z \subseteq G$:

(i) $(Xy \cdot z)X = X(y \cdot zX)$.

(ii) $(xY \cdot z)x = x(Y \cdot zx)$.

(iii) $(xy \cdot Z)x = x(y \cdot Zx)$.

(iv) $(XY \cdot z)X = X(Y \cdot zX)$.

(v) $(Xy \cdot Z)X = X(y \cdot ZX)$.

(vi) $(xY \cdot Z)x = x(Y \cdot Zx)$.

(vii) $(XY \cdot Z)X = X(Y \cdot ZX)$.

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Corollary

If $(G, \cdot, /, \backslash, e)$ is an extra polyloop-1, then the following are true in $(G, \cdot, /, \backslash, e)$ for all $x, y \in G$; $X, Y \subseteq G$

(i) $xy \cdot x = x \cdot yx$

(ii) $Xy \cdot X = X \cdot yX$

(iii) $xY \cdot x = x \cdot Yx$

(iv) $XY \cdot X = X \cdot YX$

Remark

If the extra polyloop-1 $(G, \cdot, /, \backslash, e)$ in the Corollary is an extra loop, then the flexibility law in (i) above is also true. The identities in the Corollary, (i) - (iv) will jointly be called flexibility laws for extra polyloop-1.

Theorem

Let $(G, \cdot, /, \backslash, e)$ be an extra polyloop-1, then the following are true:
(i) $(G, \cdot, /, \backslash, e)$ is an inverse property polyloop (i.e. $a \in ax \cdot x^\rho$ and $b \in x^\lambda \cdot xb \ \forall \ a, b, x \in G$).
(ii) $a \in aX \cdot X^\rho$ and $b \in X^\lambda \cdot Xb \ \forall a, b \in G; X \subseteq G$.
(iii) $Z \subseteq Zx \cdot x^\rho$ and $Y \subseteq x^\lambda \cdot xY \ \forall \ x \in G$ and $Y, Z \subseteq G$.
(iv) $Q \subseteq QX \cdot X^\rho$ and $P \subseteq X^\lambda \cdot XP \ \forall Q, P, X \subseteq G$.

Proof

(i) Suppose $(G, \cdot, /, \backslash, e)$ is an extra polyloop-1, then we have that:

$$(xy \cdot z)x = x(y \cdot zx) \tag{1}$$

Let $e \in zx \implies z \in e/x = \{x^\lambda\} \implies z = x^\lambda$ then equation (1) becomes:

Proof Contd.

$$(xy \cdot x^{\lambda})x = x(y \cdot x^{\lambda}x).$$

$$\implies \qquad xy \subseteq (xy \cdot x^{\lambda})x \tag{2}$$

Let $a \in xy \qquad \implies y \in x\backslash a$ then (2) becomes:

$$x(x\backslash a) \subseteq (x(x\backslash a) \cdot x^{\lambda})x$$

$$\implies a\bar{L}_x L_x \subseteq (a\bar{L}_x L_x \cdot x^{\lambda})x$$

$$\implies a \in ax^{\lambda} \cdot x \implies a \in a(x^{\rho})^{\lambda} \cdot x^{\rho}$$

$$a \in ax \cdot x^{\rho} \tag{3}$$

Next, let $e \in xy \implies y \in x\backslash e = \{x^{\rho}\} \implies y = x^{\rho}$
Putting this in (1) we have:

$$(xx^{\rho} \cdot z)x = x(x^{\rho} \cdot zx)$$

$$\implies zx \subseteq x(x^{\rho} \cdot zx) \tag{4}$$

Proof Contd.

Let $b \in zx \implies z \in b/x$, then (4) becomes:

$$(b/x)x \subseteq x(x^\rho \cdot (b/x)x)$$

$$\implies b\bar{R}_x R_x \subseteq x(x^\rho \cdot b\bar{R}_x R_x)$$

$$\implies b \in x \cdot x^\rho b \implies b \in x^\lambda \cdot (x^\lambda)^\rho b.$$

$$b \in x^\lambda \cdot xb \tag{5}$$

Hence, by (3) and (5), we have that $(G, \cdot, /, \backslash)$ is an inverse property polyloop.
The proofs of $(ii) - (iv)$ follow.

Proof Contd.

Putting $Z = P\bar{R}_X$ in the last relation, we have:

$$P\bar{R}_X R_X \subseteq X(X^\rho \cdot \bar{R}_X R_X)$$

$$P \subseteq X \cdot X^\rho P \implies P \subseteq X^\lambda \cdot (X^\lambda)^\rho P$$

$$\implies P \subseteq X^\lambda \cdot XP$$

Corollary

If $(G, \cdot, /, \backslash, e)$ is an extra polyloop-1, then:
(i) $a^\rho = a^\lambda = \bar{a}$ ($i.e.\ J_\rho = J_\lambda = J$) $\forall\ a \in G$
(ii) $A^\rho = A^\lambda = \bar{A}\ \forall\ A \subseteq G$

Proof

This follows from the Theorem.

Autotopism of extra Polyloop-1

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Theorem

Let $(G, \cdot, \backslash, /, e)$ be a polyloop, then $(G, \cdot, \backslash, /, e)$ is an extra polyloop-1 if and only if $(L_X, \bar{R}_X, L_X\bar{R}_X)$ is an autotopism of $(G, \cdot, \backslash, /, e)$ for all $X \subseteq G$.

Proof

Let $(G, \cdot, \backslash, /, e)$ be a polyloop, then $(G, \cdot, \backslash, /, e)$ is an extra polyloop-1 iff

$$(Xy \cdot z)X = X(y \cdot zX)$$

$$\implies (yL_X \cdot z)R_X = (y \cdot zR_X)L_X.$$

Let $A_1 = (yL_X \cdot z)R_X$ and $A_2 = (y \cdot zR_X)L_X$.

Conclusion and Further Studies

It is worthy of note that the generalization of classical quasigroup and loop is achieved by axiomatizing Polyquasigroup and polyloop. It was evident that polyquasigroup and polyloop are birthed from a non-commutative quasigroup and loop respectively.

A general sense of constructing extra polyloop, characterized by seven equivalent identities will be investigated.

In future work, the extra polyloop will be investigated for other analogous algebraic properties that were satisfied by the extra loop in the classical sense. and this will further be investigated with and illustrated with examples.

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