When is the commutant of a Moufang loop normal?

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Loops '23, Będlewo

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Is it normal?

(Basic) Inner Mappings, Normality

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A subloop is *normal* if it is invariant, as a set, under the (basic) inner mappings.

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Nucleus:

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Conjecture (Doro):

If $\mathrm{Nuc}(L) = 1$, then $\mathrm{C}(L)$ is normal.

Doro

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Simple Moufang loops

BY STEPHEN DORO
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Introduction. If H is a Moufang loop, and $x \in H$, there are defined permutations of H, $L(x): y \mapsto xy$ and $R(x): y \mapsto yx$. The group Gr (H), generated by these permutations for all choices of x, is called the multiplication group of H. It has a close connexion with the structure of H, as shown, for instance, in the papers of Albert(1). The purpose of this paper is to investigate the correspondence between groups and loops, so that group theoretic results may be applied to determine the structure of Moufang loops.

Glauberman(6) points out that for Moufang loops M with trivial nucleus, the multiplication group Gr (M) admits a certain dihedral group D of automorphisms, with $|D| = 6$. For fixed generators $\sigma, \rho \in D$ with $|\sigma| = 2$, $|\rho| = 3$, the following equation holds for any x in Gr (M):

$$[x,\sigma]\,[x,\sigma]^{\rho}\,[x,\sigma]^{\rho^2} = 1, \text{ i.e. } ([x,\sigma]\rho^2)^3 = 1. \quad (1)$$

We shall call groups G with such automorphisms, and satisfying the condition $[G, D] = G$, groups with triality, since the most striking example is $D_4(q)$ with its graph automorphisms.

History, Drama

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Classification and conjecture.

Counterexamples, lots of them

A. Grishkov and A. Zavarnitsine. “Moufang loops with nonnormal commutative centre”, *Math. Proc. Cambridge Philos. Soc.*, 2021.

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Two different infinite families of counterexamples. For any Moufang loop of exponent 3 which does not satisfy the identity $[[x, y], z] = 1$ there is a Moufang loop with nonnormal commutant. One family uses groups with triality. Smallest example has order $3^6 = 729$.

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Theorem

In a Moufang loop, L, if any one of x, y and z is in $\mathrm{C}(L)$, then $zR(x, y)^3 = z$.

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Thus, the question of the normality of $\mathrm{C}(L)$ becomes: is $d \in \mathrm{C}(L)$? Explicitly: if e is an arbitrary element in L, does $d \cdot e = e \cdot d$?

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Fix $a, b, e \in L$ *and* $c \in \mathrm{C}(L)$*; set* $d = cR(a,b) = (ca \cdot b)(b^{-1}a^{-1})$.

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1. *If* a, b, *or* c *is a cube, then* $d = c$.
2. *If* e *is a product of cubes, then* $d \cdot e = e \cdot d$.

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1. *If a, b, or c is a cube, then $d = c$.*
2. *If e is a product of cubes, then $d \cdot e = e \cdot d$.*

Corollary

If L is generated by cubes, then $\mathrm{C}(L)$ is normal.

Generalized Conjecture

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Generalized Conjecture:
"$\mathrm{Nuc}(L) = 1$" lives or dies with "$\mathrm{Mlt}(L)$ admits triality."

Dziękuję!

Thanks for your kind attention!