Supernilpotent loops

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The structure of finite nilpotent groups

Classical Theorems: [quite easy to prove]

- Groups of prime power order are nilpotent.
- G is a finite nilpotent group $\Rightarrow G \simeq \prod G_p$ where G_p is a nilpotent group of order power of p

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Folklore knowledge: Both properties, in general, fail in loops.

Bad loops

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	d	1	a	c
c	c	b	d	1	a
d	d	c	a	b	1

not nilpotent

Q_1 is simple, $Z(Q_1) = 1$

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	4	5	6	1	2
4	4	3	6	5	2	1
5	5	6	2	1	3	4
6	6	5	1	2	4	3

nilpotent, no primary decomposition

$Z(Q_2) = \{1, 2\}$ is the only proper normal subloop
$\mathrm{Mlt}(Q_2)$ is not nilpotent, has order 24
$\mathrm{Inn}(Q_2) = (\mathrm{Mlt}(Q_2))_1$ is an abelian group of order 4

Ad hoc idea

Theorem: [Bruck 1940s]
$\mathrm{Mlt}(Q)$ is nilpotent $\Rightarrow$ Q is centrally nilpotent

Theorem: [Wright 1969]
Q finite, $\mathrm{Mlt}(Q)$ is nilpotent $\Rightarrow$ $Q \simeq \prod Q_p$ where Q_p is a nilpotent loop of order power of p

... is this a better notion of nilpotence for loops?

Systematic idea

UNIVERSAL ALGEBRA

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Terrible idea: CATEGORY THEORY [joke suggested by @ProfKinyon]

What universal algebra did for us?

abelian	$\longleftrightarrow$	abelian group
	$\Downarrow$	
???	$\longleftrightarrow$	$\prod Q_p$ for Q_p nilpotent p-loop
	$\Downarrow$	
nilpotent	$\longleftrightarrow$	centrally nilpotent
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solvable	$\longleftrightarrow$	[S, Vojtěchovský 2014]
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??? $=$ *supernilpotence*

Commutator, nilpotence

$C(\alpha;\beta;\delta)$ iff for every term t and every $\bar{a} \stackrel{\alpha}{\equiv} \bar{b}$, $\bar{u} \stackrel{\beta}{\equiv} \bar{v}$

$$t(\bar{a},\bar{u}) \stackrel{\delta}{\equiv} t(\bar{a},\bar{v}) \;\Rightarrow\; t(\bar{b},\bar{u}) \stackrel{\delta}{\equiv} t(\bar{b},\bar{v})$$

The *commutator* $[\alpha,\beta]$ is the smallest δ such that $C(\alpha;\beta;\delta)$.

The *center* $\zeta(A)$ is the largest congruence ζ such that $C(\zeta;1_A;0_A)$.

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An algebra A is k-nilpotent, if there are congruences α_i such that

$$0_A = \alpha_0 \leq \alpha_1 \leq ... \leq \alpha_k = 1_A$$

and $\alpha_{i+1}/\alpha_i \leq \zeta(A/\alpha_i)$.

Fact: A loop is k-nilpotent if and only if centrally nilpotent of class $\leq k$.

Higher commutator, supernilpotence

$C_n(\alpha_1, ..., \alpha_{n-1}; \beta; \gamma)$ iff for every term t and every $\bar{a}_i \overset{\alpha_i}{\equiv} \bar{b}_i$, $\bar{u} \overset{\beta}{\equiv} \bar{v}$

$$t(\bar{x}_1, ..., \bar{x}_n, \bar{u}) \overset{\delta}{\equiv} t(\bar{x}_1, ..., \bar{x}_n, \bar{v}) \qquad \forall (\bar{x}_1, ..., \bar{x}_n) \in \{\bar{a}_1, \bar{b}_1\} \times ... \times \{\bar{a}_n, \bar{b}_n\} \neq \{(\bar{b}_1, ..., \bar{b}_n)\}$$

$$\Downarrow$$

$$t(\bar{b}_1, ..., \bar{b}_n, \bar{u}) \overset{\delta}{\equiv} t(\bar{b}_1, ..., \bar{b}_n, \bar{v}).$$

The *n-ary commutator* $[\alpha_1, ..., \alpha_n]$ is the smallest δ such that $C_n(\alpha_1, \ldots, \alpha_{n-1}; \alpha_n; \delta)$.

Fact: $[\alpha_1, ..., \alpha_n] \geq [\alpha_1, [\alpha_2, [..., [\alpha_{n-1}, \alpha_n]]]]$ (in Mal'tsev varieties)

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An algebra is *k-supernilpotent* if $[1_A, ..., 1_A] = 0_A$.

Supernilpotence – a better “definition”

Theorem: [Aichinger, Mudrinski, 2010]
In Mal'tsev varieties,

1. an algebra is k-supernilpotent if and only if all absorbing polynomials of arity $> k$ are constant.
2. a finite algebra is k-supernilpotent if and only if $A \simeq \prod A_p$ where A_p is a nilpotent algebra of order power of p

In loops:

Polynomial in Q is a term with constants from Q.

Example: $p(x, y, z) = (x/a)(y\backslash(za))$ where $a \in Q$ is fixed

A polynomial is *absorbing* if $p(a_1, ..., a_n) = 1$ whenever at least one $a_i = 1$.

Examples: $[x, y]$, $[x, y, z]$, $L_{x,y}(z)/z$, ..., $[xy, u]/([x, u][y, u])$, ...

Supernilpotent groups

Theorem: [Aichinger, Ecker, 2006; S, Vojtěchovský 2023]
A group is k-supernilpotent iff k-nilpotent.

In general, not at all.

1. k-supernilpotence $\Rightarrow$ k-nilpotence
2. nilpotence $\not\Rightarrow$ supernilpotence
3. the degree of supernilpotence can be $>>$ degree of nilpotence

Degrees of nilpotence

- $\mathrm{cl}_{cn}(Q)$ = the class of central nilpotence of Q
- $\mathrm{cl}_{m}(Q)$ = the class of nilpotence of $\mathrm{Mlt}(Q)$
- $\mathrm{cl}_{sn}(Q)$ = the class of supernilpotence of Q

Theorem [S+Semanišinová - Bruck - Wright]:

$$\mathrm{cl}_{sn}(Q) \geq \mathrm{cl}_{m}(Q) \geq \mathrm{cl}_{cn}(Q)$$

$$Q \text{ finite } \Rightarrow [\mathrm{cl}_{sn}(Q) < \infty \Leftrightarrow \mathrm{cl}_{m}(Q) < \infty]$$

Examples:

- the bad loop of *order 6*: $\mathrm{cl}_{cn}(Q) = 2$, $\mathrm{cl}_{m}(Q) = \mathrm{cl}_{sn}(Q) = \infty$
- 34 loops of *order 8*: $\mathrm{cl}_{cn}(Q) = 2$, $\mathrm{cl}_{m}(Q) = 3$, $\mathrm{cl}_{sn}(Q) \geq 4$

Equational basis for k-supernilpotence, in groups

In groups: supernilpotence = nilpotence

A group is k-(super)nilpotent if and only if

$$[x_1, [x_2, [...., [x_k, x_{k+1}]]]] = 1$$

Equational basis for k-supernilpotence, in loops

Let $[\![x, y]\!]$ and $[\![x, y, z]\!]$ be any terms such that, in all loops,

$$[\![x, y]\!] = 1 \iff xy = yx$$
$$[\![x, y, z]\!] = 1 \iff x(yz) = (xy)z$$

Example: the standard commutator and associator

$$[\![x, y]\!] = (yx)\backslash(xy), \; [\![x, y, z]\!] = x(yz)\backslash(xy)z$$

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Easy facts:

1-supernilpotence: $[\![x,y]\!] = [\![x,y,z]\!] = 1$ (abelian groups)

2-supernilpotence: $[\![x,[\![y,z]\!]]\!] = [\![x,y,z]\!] = 1$ (2-nilpotent groups)

Equational basis for 3-supernilpotent loops

Theorem: [S, Vojtěchovský, 2023]

TFAE for a loop Q:

- Q is 3-supernilpotent
- Q satisfies the following identities for all $[\![.,.]\!]$, $[\![.,.,.]\!]$
- Q satisfies the following identities for the standard $[\![.,.]\!]$, $[\![.,.,.]\!]$

$$1 = [\![x, [\![y, u, v]\!]]\!] \tag{1}$$

$$1 = [\![x, y, [\![u, v, w]\!]]\!] = [\![x, [\![u, v, w]\!], y]\!] = [\![[\![u, v, w]\!], x, y]\!] \tag{2}$$

$$1 = [\![x, y, [\![u, v]\!]]\!] = [\![x, [\![u, v]\!], y]\!] = [\![[\![u, v]\!], x, y]\!] \tag{3}$$

$$1 = [\![x, [\![y, [\![u, v]\!]]\!]]\!] = [\![x, [\![[\![u, v]\!], y]\!]]\!] \tag{4}$$

$$1 = [\![[\![y, [\![u, v]\!]]\!], x]\!] = [\![[\![[\![u, v]\!], y]\!], x]\!] \tag{5}$$

$$1 = [\![[\![x, y]\!], [\![u, v]\!]]\!] \tag{6}$$

$$[\![xy, u, v]\!] = [\![x, u, v]\!]\,[\![y, u, v]\!] \tag{7}$$

$$[\![u, xy, v]\!] = [\![u, x, v]\!]\,[\![u, y, v]\!] \tag{8}$$

$$[\![u, v, xy]\!] = [\![u, v, x]\!]\,[\![u, v, y]\!] \tag{9}$$

Problems

- Find a functions f, g such that

$$\mathrm{cl}_{sn}(Q) \leq f(cl_{cn}(Q)), \qquad \mathrm{cl}_{sn}(Q) \leq g(cl_{m}(Q))$$

  for every supernilpotent loop Q.

- Is the implication "$\mathrm{Mlt}(Q)$ nilpotent $\Rightarrow$ Q supernilpotent" true for infinite loops?

- Finite equational basis for the variety of k-supernilpotent loops.