Half-automorphism group of a class of Bol loops

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Introduction

- Given a loop L, a bijection $f : L \to L$ is a *half-automorphism* if

$$f(xy) \in \{f(x)f(y), f(y)f(x)\},$$

  for every $x, y \in L$. We say that a half-automorphism is *proper* if it is neither an automorphism nor an anti-automorphism.

- In the year 1957, W.R. Scott proved that there is no proper half-homomorphism between two groups.

- He also gave an example of a loop of order 8 that has a proper half-automorphism, so Scott's result can not be generalized to all loops.

Preliminaires

- We are interested in the half-automorphism group of a certain Bol loop.
- If L is a Bol loop and if it has an anti-automorphism, then L is a Moufang loop.
- If φ is a half-automorphism of a Bol loop L that is not Moufang, then φ is either a proper half-automorphism or an automorphism of L.

Construction of a class of Bol loops

Proposition (Foguel, Kinyon, Phillips)

Let G be a group, $H \leq G$, and $B \subset G$ a right transversal of H in G. If B is a twisted subgroup of G, then B with the operation

$$x \cdot y = z, \text{ if } xy = hz, \text{ for some } h \in H, \tag{1}$$

is a Bol loop. Conversely, if H is core-free and $(B, \cdot)$ is a Bol loop, then B is a twisted subgroup of G.

Construction of a class of Bol loops

- Let M be an abelian group. The *generalized dihedral group of M* can be defined by $D(M) = M \cup Mr$, where $r \notin M$, $r^2 = 1$ and $rxr = x^{-1}$, for every $x \in M$.
- We have that $H = 0 \times 0 \times \{1, r\}$ is a subgroup of order 2 of the direct product $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times D(M)$.
- The set
$$B = \{(0,0,x), (l,s,rx) \,|\, x \in M,\ l,s \in \mathbb{Z}_2, (l,s) \neq (0,0)\}.$$
is a right transversal of H in G which contains $(0,0,1)$, the identity of G. Also, B is a twisted subgroup of G.

Construction of a class of Bol loops

- The set B is a Bol loop with the following operation

$$
\begin{aligned}
(0,0,x)\cdot(0,0,y) &= (0,0,xy),\\
(0,0,x)\cdot(l,s,ry) &= (l,s,rx^{-1}y),\\
(l,s,rx)\cdot(0,0,y) &= (l,s,rxy),\\
(1,1,rx)\cdot(1,1,ry) &= (0,0,x^{-1}y),\\
(l,s,rx)\cdot(u,v,ry) &= (l+u,s+v,rx^{-1}y).
\end{aligned}
$$

Construction of a class of Bol loops

- Let M be an abelian group. Define $L_M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times M$ and consider the following operation on L_M:

$$(l, s, x) * (u, v, y) = \begin{cases} (l, s, xy), & \text{if } u = v = 0, \\ (l + u, s + v, x^{-1}y), & \text{otherwise.} \end{cases}$$

- $\psi : (L_M, *) \to (B, \cdot)$; $\psi((0, 0, x)) \mapsto (0, 0, x)$ and $\psi((l, s, x)) \mapsto (l, s, rx)$, where $(l, s) \neq (0, 0)$. is an isomorphism, and hence $(L_M, *)$ is a Bol loop.

Construction of a class of Bol loops

Proposition

If M is an elementary abelian 2-group, then $(L_M, *)$ is also an elementary abelian 2-group. If M is not an elementary abelian 2-group, then $(L_M, *)$ is a nonassociative, noncommutative Bol loop.

Corollary

If M is an abelian group with exponent greater than 2, then $(L_M, *)$ is not a Moufang loop. In particular, L_M has no anti-automorphisms.

Half-automorphisms of the Bol loop L_M

- Let M be a finite abelian group with exponent greater than 2 and let L_M be the Bol loop constructed previously.

- Let us denote L_M as $L_M = K \times M$, where $K = \{1, a, b, c\}$ is the Klein group and the multiplication in L_M will be given by

$$
\begin{array}{rcccl}
(1, x) & * & (1, y) & = & (1, xy) \\
(A, x) & * & (1, y) & = & (A, xy) \\
(1, x) & * & (B, y) & = & (B, x^{-1}y) \\
(A, x) & * & (B, y) & = & (AB, x^{-1}y),
\end{array}
$$

where $A, B \neq 1$.

Half-automorphisms of the Bol loop L_M

- Recall that $f : L_M \to L_M$ is a half-automorphism of L_M if f is a bijection of L_M and $f(XY) \in \{f(X)f(Y), f(Y)f(X)\}$, for every X and Y in L_M.

Proposition

Let $(1, M) = \{(1, x) \in L_M; x \in M\}$ and let f be a half-automorphism of L_M. Then $f(1, M) = (1, M)$.

Proposition

Let f be a half-autmorphism of L_M. For every $x \in M$, consider $f^{''}(x) \in M$ as $(1, f^{''}(x)) = f(1, x)$. Then $f'' : M \to M$ is an automorphism of M.

Half-automorphisms of the Bol loop L_M

- Since the set $(K, 1)$ is a subgroup of L_M isomorphic to K, $f(K, 1)$ is a subloop of L_M of order 4, and it is a group isomorphic to K.
- Let $\mathcal{H}_M = \{H \leq L_M \; ; \; H \cong K, |H \cap (1, M)| = 1\}$.

Proposition

Let f be a half-automorphism of L_M. Then $f(K, 1) \in \mathcal{H}_M$.

Half-automorphisms of the Bol loop L_M

Proposition

Let $H \in \mathcal{H}_M$. There are three possibilities:
(i) $H = (K, 1)$,
(ii) $H = \{(1,1),(A,x),(B,x),(C,1)\}$, *with* $\{A,B,C\} = \{a,b,c\}$ *and* $o(x) = 2$,
(iii) $H = \{(1,1),(A,x),(B,y),(C,xy)\}$, *with* $\{A,B,C\} = \{a,b,c\}$ *and* $x \neq y$ *are elements of order equal to* 2.
In particular, if M *has odd order, then* $\mathcal{H}_M = \{(K,1)\}$.

Half-automorphisms of the Bol loop L_M

Corollary

Consider M as a finite abelian group of even order and exponent greater than 2 and write $M = C_{2^{i_1}} \times C_{2^{i_2}} \times \ldots \times C_{2^{i_s}} \times M_1$, where $|M_1|$ is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$. Then $|\mathcal{H}_M| = 4^s$.

Half-automorphisms of the Bol loop L_M

Corollary

Let f be a half-automorphism of L_M. Then there exist a unique $f' \in Aut(K)$ and $x, y \in M$ such that $o(x), o(y) \leq 2$ and

$$f(A,1) = (f'(A), \alpha_{(x,y)}(A)), \text{ for every } A \in K,$$

where $\alpha_{(x,y)}(1) = 1$, $\alpha_{(x,y)}(a) = x$, $\alpha_{(x,y)}(b) = y$ and $\alpha_{(x,y)}(c) = xy$.

Remark

The mapping $\alpha_{(x,y)} : K \to M$ is a homomorphism and $(1, \alpha_{(x,y)}(A)) \in \mathcal{Z}(L_M)$, for every $A \in K$.

Half-automorphisms of the Bol loop L_M

- Every element (A, x) in L_M can be written as $(A, 1)(1, x)$;
- There are $f' \in Aut(K)$ and $u, v \in M$ such that $f(A, 1) = (f'(A), \alpha_{(u,v)}(A))$, where $o(u), o(v) \leq 2$. For $A \neq 1$,

$$f(A, x) = f((A, 1)(1, x)) \in \{(f'(A), 1)(1, f''(x)), (1, f''(x))(f'(A), 1)\},$$

and so

$$f(A, x) \in \{(f'(A), f''(x)\alpha_{(u,v)}(A)), (f'(A), f''(x^{-1})\alpha_{(u,v)}(A))\}.$$

Half-automorphisms of the Bol loop L_M

Then if $f' \in Aut(K)$, $f'' \in Aut(M)$ and $u, v \in M$ are elements such that $o(u), o(v) \leq 2$, we define $F^{+}_{(f',f'',u,v)}, F^{-}_{(f',f'',u,v)} : L_M \to L_M$ by

$$F^{+}_{(f',f'',u,v)}(A, x) = (f'(A), f''(x)\alpha_{(u,v)}(A)) \quad \text{and}$$

$$F^{-}_{(f',f'',u,v)}(A, x) = \begin{cases} (f'(A), f''(x)\alpha_{(u,v)}(A)), & \text{if} \quad A = 1, \\ (f'(A), f''(x^{-1})\alpha_{(u,v)}(A)), & \text{otherwise.} \end{cases}$$

For making the notation easier, we will write $f^{+}_{(u,v)}$ and $f^{-}_{(u,v)}$ instead of $F^{+}_{(f',f'',u,v)}$ and $F^{-}_{(f',f'',u,v)}$, respectively.

Half-automorphisms of the Bol loop L_M

Proposition

$f^+_{(u,v)}$ is an automorphism and $f^-_{(u,v)}$ is a proper half-automorphism of L_M.

Proposition

Let g be an automorphism of L_M. Then $g = g^+_{(u,v)}$.

Proposition

Let g be a proper half-automorphism of L_M. Then $g = g^-_{(u,v)}$.

Half-automorphisms of the Bol loop L_M

Theorem

Let $Half(L_M)$ be the group of half-automorphisms of L_M. Then $Half(L_M)$ is the set

$$\{F^{+}_{(f',f'',u,v)}, F^{-}_{(f',f'',u,v)} \mid f' \in Aut(K), f'' \in Aut(M), u, v \in M, o(u), o(v) \leq 2\}.$$

Furthermore:
(a) If $|M|$ is odd, then $|Half(L_M)| = 2.|Aut(K)|.|Aut(M)|$.
(b) If $|M|$ is even, then $|Half(L_M)| = 2^{2s+1}.|Aut(K)|.|Aut(M)|$, where the abelian group M is the direct product $C_{2^{i_1}} \times C_{2^{i_2}} \times ... \times C_{2^{i_s}} \times M_1$, $|M_1|$ has odd order, $s \geq 1$ and $i_j \geq 1$, for all j, and C_n denotes the cyclic group of order n.

Half-automorphism group of L_M

Proposition

$Half(L_M) \cong C_2 \times Aut(L_M)$.

Half-automorphism group of L_M

Proposition

Let $\mathcal{A} = \{F^{+}_{(f',f'',1,1)} \mid f' \in Aut(K), f'' \in Aut(M)\}$. Then $\mathcal{A} \cong Aut(K) \times Aut(M)$.

- In the case that $|M|$ is odd, then there is no element of order 2 in M.

Theorem

Let $L_M = K \times M$ be the Bol loop where K is the Klein group and M is an abelian group of odd order. Then

$$Aut(L_M) \cong S_3 \times Aut(M) \text{ and } Half(L_M) \cong C_2 \times S_3 \times Aut(M).$$

Half-automorphism group of L_M when $|M|$ is even

- Then $M = C_{2^{i_1}} \times C_{2^{i_2}} \times \ldots \times C_{2^{i_s}} \times M_1$, where M_1 is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$, for all j.
- The set $H = \{x \in M \mid o(x) \leq 2\}$ is a subgroup of M of exponent 2 with $|H| = 2^s$.
- Denote by I_K and I_M the identity mappings of K and M and consider

$$\mathcal{B} = \{F^{+}_{(I_K, I_M, x, y)} \mid x, y \in H\}.$$

  subgroup of $Aut(L_M)$.
- Also, $|\mathcal{A} \cap \mathcal{B}| = 1$, and so, $|\mathcal{A}\mathcal{B}| = |\mathcal{A}|.|\mathcal{B}| = 2^{2s}.|Aut(K)|.|Aut(M)|$

Half-automorphism group of L_M

Proposition

Let K be the Klein group, let M be an abelian group of even order and let $L_M = K \times M$ be the associated Bol loop. $Aut(L_M) = \mathcal{A}\,\mathcal{B} = \mathcal{B}\,\mathcal{A}$.

Proposition

Using the notation above,

(a) $\mathcal{B} \cong H \times H \cong C_2^{2s}$,

(b) $\mathcal{B} \lhd Aut(L_M)$ *and*

(c) $Aut(L_M)/\mathcal{B} \cong \mathcal{A}$.

Half-automorphism group of L_M

- Define $\sigma : \mathcal{A} \to \mathcal{B}$, where, for $\alpha \in \mathcal{A}$, $\sigma(\alpha) = \sigma_\alpha$ and $\sigma_\alpha(\beta) = \alpha\beta\alpha^{-1}$, for all $\beta \in \mathcal{B}$.
- $Aut(L_M) \cong \mathcal{A} \overset{\sigma}{\ltimes} \mathcal{B}$.

Theorem

Let M be a finite abelian group such that its exponent is greater than 2. Write $M = C_{2^{i_1}} \times C_{2^{i_2}} \times \ldots \times C_{2^{i_s}} \times M_1$, where M_1 is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$, for all j. Then

$$Aut(L_M) \cong \mathcal{A} \overset{\sigma}{\ltimes} \mathcal{B} \text{ and } Half(L_M) \cong C_2 \times (\mathcal{A} \overset{\sigma}{\ltimes} \mathcal{B}),$$

where $\mathcal{A} \cong S_3 \times Aut(M)$ and $\mathcal{B} \cong C_2^{2s}$.

Example 1

- Let $M = C_3$, the cyclic group of order 3. Then L_M is a nonassociative Bol loop of order 12, which is recognized by the command "RightBolLoop$(12, 3)$" in the library of loops of the LOOPS package.
- Since $Aut(M) = C_2$, we have that
$$Aut(L_M) \cong C_2 \times S_3 \text{ and } Half(L_M) \cong C_2^2 \times S_3$$
- L_M has 24 half-automorphisms, from which 12 are proper.

Example 2

- Let M be the group $C_4 \times C_2$. This group is recognized by the command "SmallGroup$(8,2)$" in GAP. The Bol loop L_M is nonassociative and has order 32.
- Since $Aut(M) = D_8$, the dihedral group of order 8, we have that

$$Aut(L_M) \cong (S_3 \times D_8) \overset{\sigma}{\ltimes} C_2^4 \text{ and } Half(L_M) \cong C_2 \times ((S_3 \times D_8) \overset{\sigma}{\ltimes} C_2^4)$$

- L_M has 1536 half-automorphisms, from which 768 are proper.

Thank you!

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