Algebraically Describing Color Trades on Complete Bipartite Graphs

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LOOPS 23

Preliminaries

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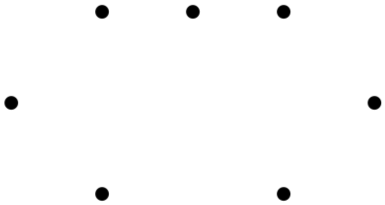

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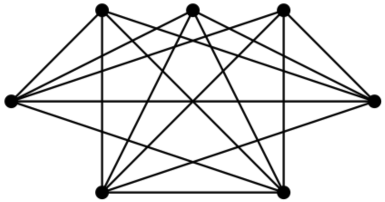

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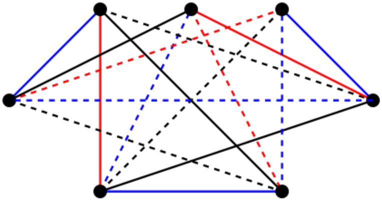

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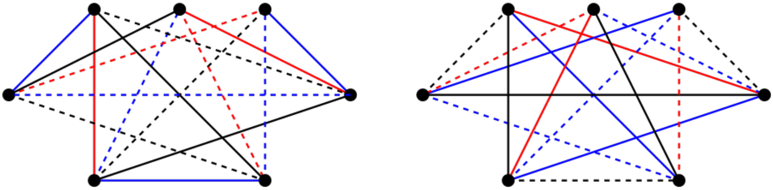

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- Essentially, partition an object into subsets satisfying some list of properties. If we can partition the object into a different set of subsets which still satisfy the same list of properties, we say the partitions form a trade.

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  1. For every $v \in V(G)$, the set of colors assigned to edges incident to v under C_1 is the same as the set of colors assigned under C_2.
  2. For every $e \in E(G)$, the color assigned to e under C_1 is different than the color assigned under C_2.

- The Color-Trade-Spectrum of G, denoted $\mathrm{CTS}(G)$, consists of all values of k for which G has a pair of mate-colorings using k colors.

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- Can we find bounds for the minimum and maximum values of the CTS of a given graph?

Lemma 2.1

Let G be a simple graph with chromatic index $\chi'(G)$.

1. If G contains a vertex of degree one, then $CTS(G) = \emptyset$.

2. In a mate-coloring of G, each color must be assigned to at least two edges of G.

3. $\chi'(G) \leq \min CTS(G)$ and $\max CTS(G) \leq \lfloor \frac{|E(G)|}{2} \rfloor$.

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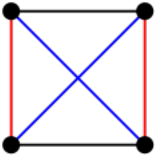
- Much of the initial work in determining color-trade-spectra consists of constructing mate k-edge-colorings which can easily be modified to yield other values in the spectrum.
- To do this, we use the following two theorems, which give us ways to respectively increase or decrease the number of values in the known color-trade-spectrum.

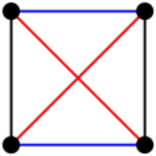
Theorem 2.3

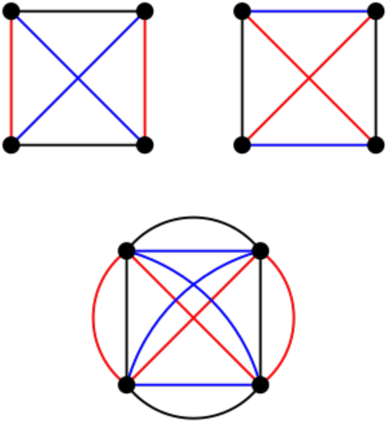
Let C_1 and C_2 be two mate k-edge-colorings of a graph G. For each j in $1 \leq j \leq k$, let H_j be the spanning subgraph of G consisting of the edges colored c_j in either C_1 or C_2. Then the components of H_j are even cycles. Let α_j be the number of these cycles and $\alpha = \sum_{j=1}^{k} \alpha_j$. Then $CTS(G)$ contains all integers in the interval $[k, \alpha]$.

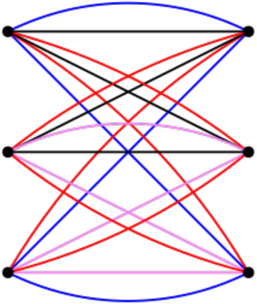
Theorem 2.4

Let C_1 and C_2 be two mate k-edge-colorings of a graph G. Denote by G' the graph whose vertices represent the k colors used in both C_1 and C_2, where vertices are adjacent if and only if the respective colors are adjacent in G under C_1 and C_2. Then $CTS(G)$ contains all integers in the interval $[\chi(G'), k]$.









Theorem 2.5

Suppose $2G$ has a k-edge-coloring where each color class is a union of pairwise vertex disjoint cycles of length greater than two. The coloring arises from a pair of mate-colorings if and only if $M(G, k)$ is bipartite.

$2 \leq m \leq n$ where m is even

- For the following theorems, we consider $K_{m,n}$ with a proper-edge-coloring C_1 along with the corresponding $n \times m$ Latin rectangle L_Δ.

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- Finding a mate-coloring for C_1 is equivalent to finding a permutation π of the entries of L with the following properties:
    1. π has no fixed points.
    2. Every color appearing in a row or column of L still appears after applying π to L.

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Theorem 4.1

$CTS(K_{m,n}) = \{n, n+1, ..., \frac{mn}{2}\}$ where $2 \leq m \leq n$ and m and is even.

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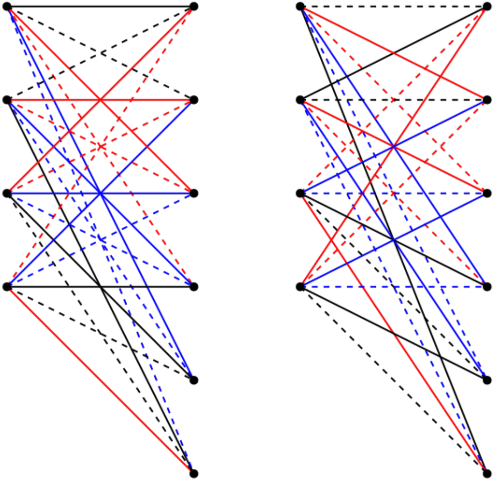
L_Δ									
	1	2	3	4	$\cdots$	$m-3$	$m-2$	$m-1$	m
	2	3	4	5	$\cdots$	$m-2$	$m-1$	m	$m+1$
	3	4	5	6	$\cdots$	$m-1$	m	$m+1$	$m+2$
	4	5	6	7	$\cdots$	m	$m+1$	$m+2$	$m+3$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-3$	$n-2$	$n-1$	n	$\cdots$	$m-7$	$m-6$	$m-5$	$m-4$
	$n-2$	$n-1$	n	1	$\cdots$	$m-6$	$m-5$	$m-4$	$m-3$
	$n-1$	n	1	2	$\cdots$	$m-5$	$m-4$	$m-3$	$m-2$
	n	1	2	3	$\cdots$	$m-4$	$m-3$	$m-2$	$m-1$

$\pi(L_\Delta)$										
	2	1	4	3	$\cdots$	$m-2$	$m-3$	m	$m-1$
	3	2	5	4	$\cdots$	$m-1$	$m-2$	$m+1$	m
	4	3	6	5	$\cdots$	m	$m-1$	$m+2$	$m+1$
	5	4	7	6	$\cdots$	$m+1$	m	$m+3$	$m+2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$n-2$	$n-3$	n	$n-1$	$\cdots$	$m-6$	$m-7$	$m-4$	$m-5$
	$n-1$	$n-2$	1	n	$\cdots$	$m-5$	$m-6$	$m-3$	$m-4$
	n	$n-1$	2	1	$\cdots$	$m-4$	$m-5$	$m-2$	$m-3$
	1	n	3	2	$\cdots$	$m-3$	$m-4$	$m-1$	$m-2$



Theorem 4.2

Let $5 \leq m \leq n$ be integers where m is odd. Then $CTS(K_{m,n}) = \{n, n+1, ..., \frac{mn}{2}\}$ if n is even and $CTS(K_{m,n}) = \{n, n+1, .., \frac{mn-1}{2}\}$ if n is odd.

Trades and Isotopies

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- Many permutations of Latin squares can be described by isotopies, which are generalizations of isomorphisms.
- $\alpha(x)\beta(y) = \gamma(xy)$

- The construction used to determine CTS($K_{n,n}$) uses $\mathbb{Z}_n$ as an initial square. The given mate for the construction can be found by the isotopy $\alpha = \gamma = \text{id}$ and $\beta = \prod_{i=1}^{\frac{n}{2}} (2i-1, 2i)$.

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- In this list, 14 out of the 35 isomorphism classes appear, and both isotopy classes appear.
- For $\mathbb{Z}_5$, 335 out of the 1,411 isomorphism classes appear, and both isotopy classes appear.

Theorem (Carr-Greer)

If Q_1 and Q_2 are isotopic quasigroups of order n where one of α, β, or γ is a derangement and the others identity maps, then the edge-colorings for $K_{n,n}$ associated with Q_1 and Q_2 form a color-trade.

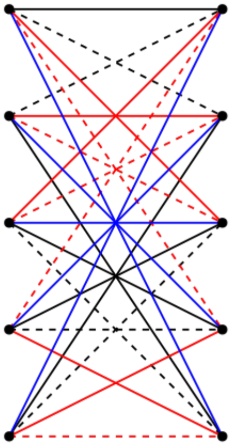
Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

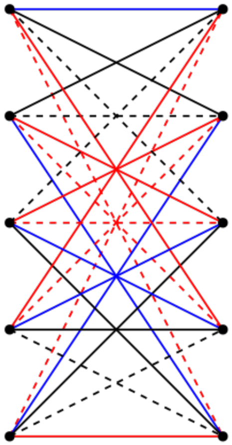
Q_3					
	2	1	4	5	3
	1	4	5	3	2
	4	5	3	2	1
	5	3	2	1	4
	3	2	1	4	5

Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_3					
	2	1	4	5	3
	1	4	5	3	2
	4	5	3	2	1
	5	3	2	1	4
	3	2	1	4	5

$$\alpha = \beta = \text{id},\ \gamma = (12)(354)$$





Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_{12}					
	2	1	4	5	3
	3	2	5	1	4
	4	3	1	2	5
	5	4	2	3	1
	1	5	3	4	2

Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_{12}					
	2	1	4	5	3
	3	2	5	1	4
	4	3	1	2	5
	5	4	2	3	1
	1	5	3	4	2

$$\alpha = \text{id},\ \beta = (25),\ \gamma = (15432)$$

Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_{12}					
	2	1	4	5	3
	3	2	5	1	4
	4	3	1	2	5
	5	4	2	3	1
	1	5	3	4	2

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Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_{118}					
	2	3	4	5	1
	3	5	2	1	4
	4	2	1	3	5
	5	1	3	4	2
	1	4	5	2	3

Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_{118}					
	2	3	4	5	1
	3	5	2	1	4
	4	2	1	3	5
	5	1	3	4	2
	1	4	5	2	3

$$\alpha = (354),\ \beta = (354),\ \gamma = (14532)$$

Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_2					
	2	1	4	5	3
	1	4	2	3	5
	5	3	1	2	4
	3	2	5	4	1
	4	5	3	1	2

Q_1					
	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

Q_2					
	2	1	4	5	3
	1	4	2	3	5
	5	3	1	2	4
	3	2	5	4	1
	4	5	3	1	2

Two nonisotopic quasigroups which correspond to color-trades.

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- While color-trades are not transitive, can we describe the ones that are purely in terms of isotopies?
- Can these techniques be applied to $K_{m,n}$? Other graphs?

Thanks!