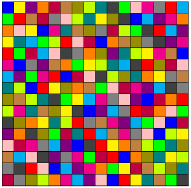
The Hadamard quasigroup product of orthogonal Latin squares



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The classical Hadamard product.

Jacques **Hadamard**

(1865-1963)

$$(A \odot B)[i,j] := A[i,j] \cdot B[i,j]$$

Hadamard product $\equiv$ Element-wise product.

Example

$A \equiv$

1	−3	2
2	0	−1
−1	2	3

$B \equiv$

0	2	−1
−1	−3	2
−1	1	3

$\Rightarrow A \odot B \equiv$

0	−6	−2
−2	0	−2
1	2	9

Generalizing the Hadamard product.

- $\text{Bin}(X) := \{\text{Binary operators on a set } X\}$

$$\begin{array}{cccccc} \text{Bin}(X) & \to & & \text{Bin}(\text{Bin}(X)) & & \\ \star & \to & \odot_\star : & \text{Bin}(X) \times \text{Bin}(X) & \to & \text{Bin}(X) \\ & & & (*, \circ) & \to & * \odot_\star \circ \end{array}$$

$$\boxed{i * \odot_\star \circ j := (i * j) \star (i \circ j)}$$

$* \equiv$

1	1	2
2	0	1
0	2	2

$\circ \equiv$

0	2	1
2	1	2
1	1	0

$\star \equiv$

0	1	2
1	2	0
2	0	1

$* \odot_\star \circ \equiv$

1	0	0
1	1	0
1	0	2

$(\text{Bin}(X), \odot_\star)$ is a groupoid.

Generalizing the Hadamard product.

Theorem (Fuchs'58)

$$\mathrm{Mult}_\star(X) := \{\circ \in \mathrm{Bin}(X) \colon (X, \star, \circ) \textit{ is a ring}\}.$$

1. $(X, \star)$ *abelian group* $\Rightarrow$ $(\mathrm{Mult}_\star(X), \odot_\star)$ *abelian group.*
2. $\mathrm{Mult}_\star(X) \cong \mathrm{Hom}\,(X,\, \mathrm{End}(X))$.

Theorem (Clay'68)

$$\mathrm{Mult}_{\star L}(X) := \{* \in \mathrm{Bin}(X) \colon (X, \star, *) \textit{ is a near-ring}\}.$$

1. $(X, \star)$ *abelian group* $\Rightarrow$ $(\mathrm{Mult}_{\star L}(X), \odot_\star)$ *abelian group.*
2. $\mathrm{Mult}_{\star L}(X) \cong \mathrm{Map}\,(X,\, \mathrm{End}(X))$.

Generalizing the Hadamard product.

Lemma

$(X, \star)$ embeds into $(\mathrm{Bin}(X), \odot_\star)$ via the homomomorphism

$$\begin{array}{ccc} (X,\star) & \hookrightarrow & (\mathrm{Bin}(X), \odot_\star) \\ c & \to & \begin{array}{rrcl} *_c : & X \times X & \to & X \\ & (a,b) & \to & a *_c b = c \end{array} \end{array}$$

Lemma

Every algebraic identity satisfied by $(X, \star)$ is also satisfied by $(\mathrm{Bin}(X), \odot_\star)$.

Example : Commutativity $(* \odot_\star \circ \equiv \circ \odot_\star *)$

$$i * \odot_\star \circ j = (i * j) \star (i \circ j) = (i \circ j) \star (i * j) = i \circ \odot_\star * j$$

- $\mathcal{Q}(X) := \{\star \in \mathrm{Bin}(X) \colon (X,\star) \text{ is a quasigroup}\}$.

Proposition

$\star \in \mathcal{Q}(X) \Leftrightarrow \odot_\star \in \mathcal{Q}\left(\mathrm{Bin}(X)\right)$.

Example :

$$(X,\star) \cong (\mathbb{Z}_2,+)$$

$$(\mathrm{Bin}(X),\odot_\star) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\cdot)$$

The Hadamard quasigroup product.

$*, \circ \in \mathrm{Bin}(X)$ are *orthogonal* ($* \perp \circ$) if, for each $(a, b) \in X \times X$, there exists exactly one pair $(x, y) \in X \times X$ such that $x * y = a$ and $x \circ y = b$.

- $\mathcal{O}(*, \circ) := \{\star \in \mathrm{Bin}(X) \colon * \perp \star \perp \circ\}$

Lemma

The map

$$\begin{array}{ccc} \mathrm{Bin}(X) & \to & \mathrm{Bin}(X) \\ \star & \to & * \odot_\star \circ \end{array}$$

is a bijection if and only if $* \perp \circ$.

The Hadamard quasigroup product.

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- $\mathcal{O}(*, \circ) := \{\star \in \mathrm{Bin}(X) \colon\ * \perp \star \perp \circ\}$

Proposition

If $* \perp \circ$, then the map

$$\begin{array}{ccc} \mathcal{Q}(X) & \rightarrow & \mathcal{O}(*, \circ) \\ \star & \rightarrow & * \odot_{\star} \circ \end{array}$$

is a bijection.

The Hadamard quasigroup product.

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- $\mathcal{O}(*,\circ) := \{\star \in \mathrm{Bin}(X) \colon * \perp \star \perp \circ\}$

Proposition

If $* \perp \circ$, then the map

$$\begin{array}{ccc} \mathcal{Q}(X) & \to & \mathcal{O}(*,\circ) \\ \star & \to & * \odot_\star \circ \end{array}$$

is a bijection.

Under which conditions $* \odot_\star \circ \in \mathcal{Q}(X)$?

The Hadamard quasigroup product.

The Hadamard quasigroup product does not preserve the Latin square property in general.

$$\star \equiv \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 1 & 2 \\ \hline 2 & 3 & 1 \\ \hline \end{array} \Rightarrow \star^2 := \star \odot_\star \star \equiv \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

Lemma

$* \odot_\circ \star \in \mathcal{Q}(X)$ *if and only if*

$$\{(x * y,\, x \circ y) :\ y \in X\}$$

and

$$\{(y * x,\, y \circ x) :\ y \in X\}$$

are Latin transversals in $\star$*, for all* $x \in X$.

The Hadamard quasigroup product.

- $\mathcal{DQ}(X) := \{\star \in \mathrm{Bin}(X) \colon\ X = \{x * x \colon\ x \in X\}\}.$

Proposition

$$\star^2 \in \mathcal{Q}(X) \Leftrightarrow \star \in \mathcal{DQ}(X)$$

$$\star \equiv \star^2 \equiv \begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline 2 & 1 & 3 \\ \hline \end{array}$$

The Hadamard quasigroup product.

- $\mathcal{DQ}(X) := \{\star \in \mathrm{Bin}(X)\colon\ X = \{x * x\colon\ x \in X\}\}$.

Proposition

$$\star^2 \in \mathcal{Q}(X) \Leftrightarrow \star \in \mathcal{DQ}(X)$$

$\star \equiv \star^2 \equiv$

1	3	2
3	2	1
2	1	3

What about successive iterations?

The Hadamard quasigroup product.

$$\odot_\ell^2\star := \odot_\rho^2\star := \star^2.$$

$$\odot_\ell^k\star := \star \odot_\star \left(\odot_\ell^{k-1}\star\right) \quad \text{and} \quad \odot_\rho^k\star := \left(\odot_\rho^{k-1}\star\right) \odot_\star \star$$

Proposition

The minimum positive integers $\ell(\star)$ and $\rho(\star)$ such that

$$\odot_\ell^{\ell(\star)+1}\star = \odot_\rho^{\rho(\star)+1}\star = \star$$

are quasigroup isomorphism invariants. They satisfy that

$$\ell(\star) = \rho(\star^t).$$

The Hadamard quasigroup product.

- $\mathcal{HQ}_\rho(X) := \{\star \in \mathcal{Q}(X) \colon \odot_\rho^k \star \in \mathcal{Q}(X), \text{ for all } k\}$.

Proposition

$$\star \in \mathcal{HQ}_\rho(X) \Leftrightarrow \left\{\left(x \odot_\rho^{k-1} \star x,\ x \star x\right) \colon x \in X\right\}$$

is a Latin transversal in $\star$, for all $k \leq \rho(\star)$.

2	4	1	3	5
1	3	5	2	4
5	2	4	1	3
4	1	3	5	2
3	5	2	4	1

$(\rho, \ell) = (3, 5)$

2	1	5	4	3
4	3	2	1	5
1	5	4	3	2
3	2	1	5	4
5	4	3	2	1

$(\rho, \ell) = (5, 3)$

2	5	3	1	4
5	3	1	4	2
3	1	4	2	5
1	4	2	5	3
4	2	5	3	1

$(\rho, \ell) = (5, 5)$

The Hadamard quasigroup product.

2	5	12	4	8	7	13	14	15	16	3	6	10	11	1	9
3	1	7	8	11	6	14	15	5	13	16	2	9	12	10	4
4	6	5	2	10	15	7	13	11	8	14	3	16	1	9	12
5	15	9	3	14	1	6	8	2	4	7	13	11	16	12	10
1	3	8	9	6	16	11	4	13	12	15	14	5	10	7	2
10	2	11	12	7	4	3	5	14	15	9	16	1	13	8	6
9	12	3	10	1	2	8	6	16	14	13	15	4	5	11	7
15	14	2	1	5	13	4	7	3	6	8	10	12	9	16	11
13	7	14	15	16	3	5	9	10	11	6	12	2	8	4	1
14	10	15	7	13	5	16	2	12	9	11	1	8	4	6	3
16	13	4	5	15	14	1	11	7	10	12	9	3	6	2	8
12	4	16	13	3	8	15	1	9	7	10	11	6	2	14	5
6	8	10	11	9	12	2	16	1	5	4	7	14	3	15	13
7	9	6	16	12	11	10	3	8	1	2	4	13	15	5	14
11	16	1	6	2	9	12	10	4	3	5	8	7	14	13	15
8	11	13	14	4	10	9	12	6	2	1	5	15	7	3	16

$$\rho = 30$$

$$\mathrm{OBin}(k,X) := \left\{(*_1,\ldots,*_k) \in (\mathrm{Bin}(X))^k : \ *_i \perp *_j, \text{for all } i,j\right\}.$$

$$\mathrm{OBin}_2(k,X) := \left\{(*_1,*_2,\star_1,\ldots,\star_{k-2}) \in \mathrm{OBin}(k,X) : \ \begin{cases} *_1,*_2 \in \mathrm{Bin}(X), \\ \star_1,\ldots,\star_{k-2} \in \mathcal{Q}(X) \end{cases}\right\}$$

Lemma

The following map is an involution.

$$\begin{array}{rccc} \Phi : & \mathrm{OBin}_2(k,X) & \to & \mathrm{OBin}_2(k,X) \\ & (*_1,*_2,\star_1,\ldots,\star_{k-2}) & \to & (*_1^\Phi,*_2^\Phi,\star_1^\Phi,\ldots,\star_{k-2}^\Phi) \end{array}$$

where

$$\begin{array}{ccccc} (x *_1 y) & *_1^\Phi & (x *_2 y) & := & x \\ (x *_1 y) & *_2^\Phi & (x *_2 y) & := & y \\ (x *_1 y) & \star_s^\Phi & (x *_2 y) & := & x *_s y, \end{array}$$

for all $s \in \{1,\ldots,k-2\}$.

The Hadamard quasigroup product of orthogonal binary operations.

		$*_1$	$*_2$	$\star_1$	$\dots$	$\star_{k-2}$
x	y	$x *_1 y$	$x *_2 y$	$x \star_1 y$	$\dots$	$x \star_{k-2} y$

$$\updownarrow$$

$*_1$	$*_2$			$\star_1$	$\dots$	$\star_{k-2}$
$x *_1 y$	$x *_2 y$	x	y	$x \star_1 y$	$\dots$	$x \star_{k-2} y$

$$\updownarrow$$

		$*_1^\Phi$	$*_2^\Phi$	$\star_1^\Phi$	$\dots$	$\star_{k-2}^\Phi$
$x *_1 y$	$x *_2 y$	x	y	$x \star_1 y$	$\dots$	$x \star_{k-2} y$

		$*_1$	$*_2$	$\star_1$	$\dots$	$\star_{k-2}$
x	y	$x *_1 y$	$x *_2 y$	$x \star_1 y$	$\dots$	$x \star_{k-2} y$

$$\updownarrow$$

$*_1$	$*_2$			$\star_1$	$\dots$	$\star_{k-2}$
$x *_1 y$	$x *_2 y$	x	y	$x \star_1 y$	$\dots$	$x \star_{k-2} y$

$$\updownarrow$$

		$*_1^\Phi$	$*_2^\Phi$	$\star_1^\Phi$	$\dots$	$\star_{k-2}^\Phi$
$x *_1 y$	$x *_2 y$	x	y	$x \star_1 y$	$\dots$	$x \star_{k-2} y$

If $*_1, *_2 \in \mathcal{Q}(X)$, then $(*_1, *_2, \star_1, \dots, \star_{k-2})$ and $(*_1^\Phi, *_2^\Phi, \star_1^\Phi, \dots, \star_{k-2}^\Phi)$ are two *paratopic* k-MOLS [Egan, Wanless'16].

$$
\begin{array}{llll}
(x *_1 y) & *_1^{\Phi} & (x *_2 y) & := \quad x \\
(x *_1 y) & *_2^{\Phi} & (x *_2 y) & := \quad y \\
(x *_1 y) & \star_s^{\Phi} & (x *_2 y) & := \quad x *_s y,
\end{array}
$$

Lemma

For each $s \in \{3, \ldots, k\}$,

$$*_1 \odot_{*_s^{\Phi}} *_2 = *_s$$

and

$$*_1^{\Phi} \odot_{*_s} *_2^{\Phi} = *_s^{\Phi}.$$

$*_1$	$*_2$	$\star_1$	$\ldots$	$\star_{k-2}$

$\updownarrow \Phi$

$*_1^{\Phi}$	$*_2^{\Phi}$	$*_1^{\Phi} \odot_{\star_1} *_2^{\Phi}$	$\ldots$	$*_1^{\Phi} \odot_{\star_{k-2}} *_2^{\Phi}$

$$\begin{array}{cccc} (x *_1 y) & *_1^{\Phi} & (x *_2 y) & := & x \\ (x *_1 y) & *_2^{\Phi} & (x *_2 y) & := & y \\ (x *_1 y) & \star_s^{\Phi} & (x *_2 y) & := & x *_s y, \end{array}$$

Lemma

For each $s \in \{3, \ldots, k\}$,

$$*_1 \odot_{*_s^{\Phi}} *_2 = *_s$$

and

$$*_1^{\Phi} \odot_{*_s} *_2^{\Phi} = *_s^{\Phi}.$$

$*_1$	$*_2$	$*_1 \odot_{\star_1} *_2$	$\ldots$	$*_1 \odot_{\star_{k-2}} *_2$

$$\updownarrow \Phi$$

$*_1^{\Phi}$	$*_2^{\Phi}$	$\star_1$	$\ldots$	$\star_{k-2}$

Theorem

If $* \perp \circ$, then the following map is a bijection.

$$\begin{array}{ccc} \mathcal{OA}(*,\circ)\cap\mathcal{Q}(X) & \rightarrow & \mathcal{OA}\left(*^{\Phi},\circ^{\Phi}\right)\cap\mathcal{Q}(X)\\ \star & \rightarrow & *^{\Phi}\odot_{\star}\circ^{\Phi} \end{array}$$

1	2	3	4	5	6	7	8
2	1	4	3	8	7	6	5
3	5	2	6	1	8	4	7
5	3	1	7	2	4	8	6
8	4	7	1	6	3	5	2
4	8	6	2	7	5	3	1
7	6	5	8	3	2	1	4
6	7	8	5	4	1	2	3

$*$

1	2	3	4	5	6	7	8
3	5	1	7	2	8	4	6
2	1	8	5	4	7	6	3
8	4	6	2	7	3	5	1
6	7	5	8	3	1	2	4
5	3	2	6	1	4	8	7
4	8	7	1	6	5	3	2
7	6	4	3	8	2	1	5

$\circ$

1	2	3	4	5	6	7	8
5	3	7	1	6	4	8	2
8	4	1	7	2	3	5	6
7	6	8	5	4	2	1	3
4	8	2	6	1	5	3	7
6	7	4	3	8	1	2	5
3	5	6	2	7	8	4	1
2	1	5	8	3	7	6	4

$\star$

1	8	7	3	2	4	6	5
8	1	2	5	7	6	4	3
5	3	1	4	8	7	2	6
2	7	4	1	6	3	5	8
3	5	8	6	1	2	7	4
4	6	5	2	3	1	8	7
6	4	3	7	5	8	1	2
7	2	6	8	4	5	3	1

$*^{\Phi}$

1	6	7	5	2	3	8	4
7	2	1	8	6	4	5	3
6	1	3	2	8	5	4	7
3	8	6	4	1	7	2	5
2	7	4	6	5	8	3	1
8	3	5	7	4	6	1	2
5	4	8	1	3	2	7	6
4	5	2	3	7	1	6	8

$\circ^{\Phi}$

1	7	4	2	3	8	5	6
6	2	5	7	8	3	4	1
5	8	3	6	4	7	1	2
7	1	2	4	6	5	8	3
4	3	8	1	5	2	6	7
3	4	1	8	7	6	2	5
8	5	6	3	2	1	7	4
2	6	7	5	1	4	3	8

$*^{\Phi}\odot_{\star}\circ^{\Phi}$

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