Moufang loops and non-commuting graphs

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Definition

A loop Q is a *Moufang loop* if for all $x, y, z \in Q$, one of the following equivalent identities hold:

$$z(x(zy)) = ((zx)z)y,$$
$$x(z(yz)) = ((xz)y)z,$$
$$(zx)(yz) = (z(xy))z,$$
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A loop Q is a *semiautomorphic inverse property loop* if for all $x, y, z \in Q$ one of the following equivalent identities hold:

$$(xy)(z \cdot xy) = (x \cdot yz)x \cdot y$$
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Theorem (Bruck)

Moufang loops are semiautomorphic inverse property loops.

Geometric Group Theory

Given a group $G = \langle S \mid R \rangle$ with generators S and relations R, constuct an associated graph usually called its *Cayley Graph* where for all $g \in G$ and $s \in S$ there is a (directed) edge from g to gs.

Goal: Study groups as geometric objects.

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Geometric Quasigroup/Loop Theory

Given a quasigroup/loop Q, construct an associated graph and study as geometric objects.

Issue: Generators and relations

Some quasigroup results

The Petersen graph is not a Cayley graph.

- Comparing lengths of cycles to the Petersen graph and generators of group "candidates".
- There has been work to show that the generalized Petersen graphs correspond to quasigroups (Mwambene).

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Question

Classify quasigroups whose Cayley graphs are Petersen type graphs.

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Some loop results

For Moufang loops

- Vojtěchovský - Moufang loop of order 12
- Stener - Moufang loops of order 16

Definition (Darafsheh)

For a non-abelian group G the non-commuting graph of G Γ_G is a graph with Vertex set $G/Z(G)$ where x, y are joined by an edge if and only if $xy \neq yx$.

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Idea

Given a loop Q construct its non-commuting graph Γ_Q.

Problem:

For a loop Q, the *commutator* $C(Q) = \{x \in Q | xy = yx \quad \forall y \in Q\}$. It is well known that in general $C(Q) \not\leq Q$.

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Normality?

Let Q be a Moufang/semiautomorphic inverse property loop. Then $C(Q)$ may not be a normal subloop.

Theorem (Chein)

Let G be a group, $g_0 \in Z(G)$, and $*$ an involutory antiautomorphism of G such that $g_0^* = g_0$ and $gg^* \in Z(G)$ for all $g \in G$. For an indeterminate t define multiplication $\circ$ on $G \cup Gt$ by

$$g \circ h = gh, \qquad g \circ ht = (hg)t, \qquad gt \circ h = (gh^*)t, \qquad gt \circ ht = g_0 h^* g,$$

where $g, h \in G$. Then $(G \cup Gt, \circ)$ is a Moufang loop. Moreover, $(G \cup Gt, \circ)$ is nonassocitive if and only if G is nonabelian.

Theorem (de Barros and Juriaans)

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Theorem

Let G be a group, $Q = (G \cup Gt, \circ)$ with $\circ$ defined in either construction. Then $C(Q) \trianglelefteq Q$.

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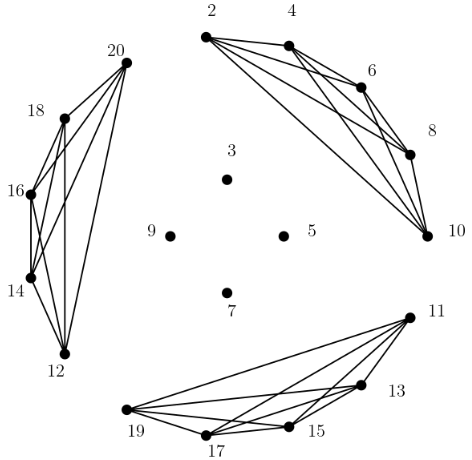
proof idea

- Show that if $a \in C(Q)$ then $a \in G$ and $a^* = a$.
- Show that $C(Q) \leq Z(Q)$.
- Verify eight cases for each construction

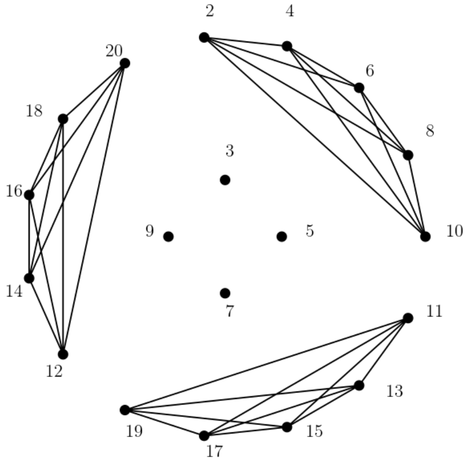
Example

Let $G = D_{10}$, $Q = (G \cup Gt, \circ)$ with Chein's $\circ$, and Γ_Q its non-commuting graph.



Example

Let $G = D_{10}$, $Q = (G \cup Gt, \circ)$ with de Barros and Juriaans' $\circ$, and Γ_Q its non-commuting graph.



Theorem (CGJ)

Let G be a group, Q_1 be $(G \cup Gt, \circ_1)$ with $\circ_1$ being Chein's, and Q_2 be $(G \cup Gt, \circ_2)$ with $\circ_2$ being de Barros and Juriaans'. Then $\Gamma_{Q_1} \cong \Gamma_{Q_2}$.

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Lemma (CGJ)

Let $G = D_{2n}$, $Q = (G \cup Gt, \circ)$, and Γ_Q its non-commuting graph. If n is odd then $\Gamma_Q \cong \overline{K}_{n-1} \nabla 3K_n$ and if n is even then $\Gamma_Q \cong \overline{K}_{n-2} \nabla (3K_n - F)$ where F is a 1-factor.

- If n is odd, then Γ_Q is a graph with $3n$ vertices each having degree $n-1$, $n-2$ neighbors, and each non-adjacent pair of vertices has 0 common neighbors.
- If n is even, then Γ_Q is a graph with n vertices each having degree $n-2$, $n-4$ neighbors, and each non-adjacent pair of vertices has $n-2$ common neighbors.

proof idea

Partition the vertices into four sets:

$$v_1 = \{a, a^2, \ldots, a^{n-1}\}, \qquad v_2 = \{b, ab, \ldots, a^{n-1}b\},$$
$$v_3 = \{t, at, \ldots, a^{n-1}t\}, \qquad v_4 = \{bt, (ab)t, \ldots, (a^{n-1}b)t\}.$$

Consider the relations such as

- $a^i \circ a^j = a^j \circ a^i$, $a^i \circ (a^j b) \neq (a^j b) \circ a^j$, $a^i \circ (a^j t) = (a^j t) \circ a^i$, $a^i \circ (a^j b)t \neq (a^j b)t \circ a^i$
- $(a^i b) \circ (a^j b) \neq (a^j b) \circ (a^i b), \ldots$

Lemma (CGJ)

Let $G = D_{2n}$, $Q = (G \cup Gt, \circ)$, and Γ_Q its non-commuting graph. If n is odd

- $\omega(\Gamma_Q) = n + 1$ (maximum size of a clique)
- $\chi(\Gamma_Q) = n + 1$ (vertex chromatic number)
- $\alpha(\Gamma_Q) = n - 1$ (maximum size of independent set)
- $\beta(\Gamma_Q) = 3n$ (minimum size of vertex covering)
- $\gamma(\Gamma_Q) = 2.$ (minimum size of dominating set)

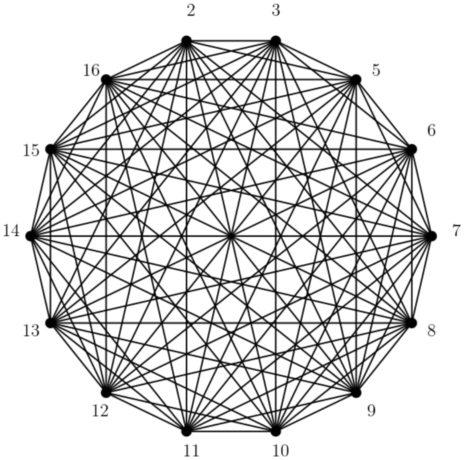
Lemma (CGJ)

Let $G = D_{2n}$, $Q = (G \cup Gt, \circ)$, and Γ_Q its non-commuting graph. If n is odd

- $\omega(\Gamma_Q) = \frac{n}{2} + 1$
- $\chi(\Gamma_Q) = \frac{n}{2} + 1$
- $\alpha(\Gamma_Q) = 6$ (for $n = 6$) and $n - 2$ else
- $\beta(\Gamma_Q) = 16$ (for $n = 6$) and $3n$ else
- $\gamma(\Gamma_Q) = 2$.

Example

Let $G = Q_8$, $Q = (G \cup Gt, \circ)$ $\circ$, and Γ_Q its non-commuting graph.



Lemma (CGJ)

Let $G = Q_{4n}$ be the generalize quaternion group, $Q = (G \cup Gt, \circ)$, and Γ_Q its non-commuting graph. Then $\Gamma_Q \cong K_{2n-2,3n\times 2}$. That is, Γ_Q consists of one independent sets of vertices of size $2n-2$ and $3n$ independent sets of vertices of size two.

Lemma (CGJ)

Let $G = Q_{4n}$, $Q = (G \cup Gt, \circ)$, and Γ_Q its non-commuting graph. Then

- $\omega(\Gamma_Q) = 3n + 1$
- $\chi(\Gamma_Q) = 3n + 1$
- $\alpha(\Gamma_Q) = 2n - 2$
- $\beta(\Gamma_Q) = 6n$
- $\gamma(\Gamma_Q) = 2$.

Isomorphism Classes

order	12	16	20	24	28	32	36	40	42	44	48	52	54
loops	1	5	1	5	1	71	4	5	1	1	51	1	2
graphs	1	2	1	3	1	10	3	3	1	1	15	1	1

order	56	60	64	81	243
loops	4	5	4262	5	72
graphs	2	5	???	2	5

Questions/TODO

- Continue to classify Γ_Q for $Q = (G \cup Gt, \circ)$ with other non-abelian groups G.
- Classify Γ_Q for Moufang loops in general.
- Consider $Q/Z(Q)$ and corresponding Γ_Q
    - Classify Γ_Q for other "well-behaved" loops Q.

THANKS!