Binary Lie Algebras with identities and diassociative loops

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Tangent algebras of smooth loops

Recall that the identities $x^2 = 0$, $J(x, y, z) = 0$, (where $J(x, y, z) = (xy)z + (yz)x + (zx)y$) are defining identities of a Lie algebra.

In 1955, A.I. Malcev, applying the Campbell-Hausdorff formula to the varieties of analytic local loops, introduced the binary Lie algebras as tangent algebras of analytic diassociative loops and Moufang-Lie algebras (now called Malcev algebras) as tangent algebras of analytic local Moufang loops.

The identities $x^2 = 0$, $J(x, y, xy) = 0$, defining the variety of binary-Lie algebras were found in the article of Gainov. On the other hand, the identities $x^2 = 0$, $J(x, y, xz) = J(x, y, z)x$, which define the variety of Malcev algebras were stated in the work of Sagle.

Tangent algebras of smooth loops. Literatura:

A.I. Malcev, Analytic loops, Mat. Sb. 36(78) (1955)(in Russian).

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Sagle, A. Malcev algebras. Trans. Amer. Math. Soc. 101 (1961)

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Definitions and notations

Let L be a set endowed with a binary operation $\cdot$.
left multiplication $L_x a = xa$
right multiplication $R_x b = bx$

$(L, \cdot, 1)$ is a **loop** if the mappings L_x, R_x are bijections, $\forall x \in L$,

$1 \cdot x = x = x \cdot 1$

left multiplication group $LMult(L)$ - group generated by $\{L_a\}_{a \in L}$

right multiplication group $RMult(L)$ - group generated by $\{R_b\}_{b \in L}$

multiplication group $Mult(L)$ - group generated by $\{L_a, R_b\}_{a,b \in L}$

inner mapping group $Int(L) = \{\phi \in Mult(L) \mid \phi(1) = 1\}$

left inner mapping group $LInt(L) = \{\phi \in LMult(L) \mid \phi(1) = 1\}$

right inner mapping group $RInt(L) = \{\phi \in RMult(L) \mid \phi(1) = 1\}$

It is known that

$$LInt(L) = \langle l_{x,y} \rangle, \quad RInt(L) = \langle r_{x,y} \rangle, \quad Int(L) = \langle l_{x,y},\, r_{x,y},\, T_x \rangle,$$

where

$$l_{x,y} = L_{(xy)}^{-1} \circ L_x \circ L_y, \quad r_{x,y} = R_{(yx)}^{-1} \circ R_x \circ R_y, \quad T_x = L_x^{-1} \circ R_x$$

As usual, a **normal subloop** is the kernel of loop homomorphism.

A subloop is normal if and only if it is invariant under inner mappings.

Bol loops, Moufang loops, Bruck loops, left automorphic loops, automorphic loops

left Bol identity: $L_x \circ L_y \circ L_x = L_c$

right Bol identity: $R_x \circ R_y \circ R_x = R_d$

A loop which is a left and a right Bol loop simultaneously is a **Moufang** loop.

$$(xy)(zx) = (x(yz))x \quad \text{(Moufang identity)}$$

It is well known that a Moufang loop is diassociative, i.e. every two elements of a loop generate a subgroup.
A (left) Bol loop L with the **automorphic inverse property**:

$$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$$

is called a (left) **Bruck loop**
A loop L with the property:
$LInt(L) \subset Aut(L)$ is called **left automorphic** and with the property:
$Int(L) \subset Aut(L)$ is called **automorphic**
Note Left Bruck loops are left automorphic loops.

Tangent algebras of the smooth Loops

The general defining identities of a tangent algebras of an analytic loops were found by L.V. Sabinin and P.O.Mikheev in 80ties. Now such algebras are called *Sabinin algebras.* Also they developed the theory of smooth Bol loops and their tangent algebras.
It is known that smooth Bruck loops describe the Symmetric Spaces and their tangent algebras are Lie triple systems.

Literatura

L.V. Sabinin, P.O. Mikheev, The differential geometry of Bol loops, Dokl. Akad. Nauk SSSR 281(5) (1985) (in Russian).

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Reductive homogeneous spaces

In 70-ties the theory of smooth left automorphic loops with additional properties and corresponding tangent algebras (which lead to reductive homogeneous spaces) was developed by M. Kikkawa and later independently by L.Sabinin.
Recall that
The homogeneous space $M \cong G/H$ is said to be reductive if there is a decomposition as a direct sum of vector spaces $g = h \oplus m$ such that $[h, m] \subseteq m$.
Here the Lie group G is acting transitevely on M, H its subgroup of isotropy and g and h are correspondent Lie algebras.
Geometrically a $C^\infty -$ manifold M with the affine connection ∇ is reductive if $\nabla R = 0$ and $\nabla T = 0$, where R and T are curvature and torsion.

Lie- Yamaguti algebra

The following algebras describe the reductive homogeneous spaces.
Definition A Lie -Yamaguti algebra is a vector space m endowed with a bilinear operation $\cdot : m \times m \to m$ and a trilinear operation $[,,] : m \times m \times m \to m$, such that for all $x, y, z, u, v, w \in m$

(1) $x^2 = 0$,
(2) $[x, x, y] = 0$,
(3)$\Sigma_{(x,y,z)}([x, y, z] + (x \cdot y) \cdot z) = 0$,
(4)$\Sigma_{(x,y,z)}[x \cdot y, z, u] = 0$,
(5) $[x, y, u \cdot v] = [x, y, u] \cdot v + u \cdot [x, y, v]$,
(6) $[x, y[u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]]$,

where $\Sigma_{(x,y,z)}$ is a cyclic sum on x, y, z.

Tangent algebras of the smooth left-automorphic Moufang Loops

Malcev algebra: $x^2 = 0, \;\; J(x, y, xz) = J(x, y, z)x.$

The tangent algebra of a smooth automorphic Moufang loop is a Lie algebra, i.e. the following identities hold:$x^2 = 0, \;, J(x, y, z) = 0$

The tangent algebra of a smooth left-automorphic Moufang loop is a Malcev algebra with the following additional identity

$$J(x, y, zw) = 0. \tag{1}$$

It is clear that this algebra obeys the identities:

$$x^2 = 0, \;\; J(x, y, xz) = J(x, y, z)x = 0. \tag{2}$$

Literatura:

Carrillo-Catalán, Ramiro; SL On smooth power-alternative loops. Comm. Algebra 32 (2004), no. 8.

Tangent algebras of type I and II

Let us call the Malcev algebras satisfying (1) ($J(x, y, zw) = 0$) Malcev algebras of first type
and the Malcev algebras satisfying (2) ($J(x, y, xz) = 0$) Malcev algebras of second type.

Theorem

The varieties of Malcev algebras of first type and of second type are different.

Example
There exists an example of a 23-dimensional algebras of second type which is not an algebra of first type.
Open Question If 23 is a minimal dimension for example of this kind?

NB There exists the free algebra of the second type, which is not of first type of dimension 29.

On Malcev algebras of type II

Question What kind of local Moufang loop does correspond to the Malcev algebras of second type? We answered the question.
Let L be a loop, such that every three elements of L generate a left automorphic subloop. We will call a loop L with this property an *almost left automorphic* loop.

Theorem A tangent algebra of a smooth almost left automorphic Moufang loop is a Malcev algebra of second type (i.e. with defining identity $J(x, y, zx) = 0$)
The local Moufang loop which corresponds to the Malcev algebra of second type is almost left automorphic.

Literatura:

Carrillo-Cataln, Ramiro; Rasskazova, Marina; LS *Malcev algebras corresponding to smooth almost left automorphic Moufang loops.* J. Algebra Appl. 17 (2018)

The speciality of Malcev algebras of type I

Malcev algebras are related to alternative algebras in the same way that Lie algebras are related to associative algebras: if A is an alternative algebra, then the algebra $A^{(-)}$, with the multiplication $[a; b] = ab - ba$, is a Malcev one. It was an open problem during many years whether the analog of the Poincare -Birkhoff -Witt theorem is true for them; that is, whether any Malcev algebra is isomorphic to a sub- algebra of an algebra $A^{(-)}$, for a suitable alternative algebra A.
Such a Malcev Algebras are called *special* in literature.

In 2017 IP Shestakov announced of the existing of the example of non-special Malcev algebra.
Conjecture Malcev algebras with the identity$J(x, y, zt) = 0$ are special.

Global Moufang loops 1

From third Sophus Lie theorem it is known that for any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ there exists a local Lie group G, such that its tangent algebra is isomorphic to $\mathfrak{g}$. Elie Cartan developed Lie theory and proved the existence of global group Lie $\tilde{G}$ with tangent algebra $\mathfrak{g}$. In the same way F.S.Kerdman developed the theory of Moufang loops and Malcev algebras. It was shown that for finite dimensional Malcev algebra $\mathfrak{m}$ there exists global Moufang loop M, such that its tangent algebra is $\mathfrak{m}$. We proved that for the subclasses of analytic Moufang loops the question of the existence of a global left automorphic Moufang loop which corresponds to the given Malcev algebra of the first type is solved positevely.

Global Moufang loops 2

To respect to the class of smooth left automorphic loops we would like to generilize our result:
Let $\mathcal{M}A_n$ be a variety of Malcev algebras with the identity $J(x_1x_2...x_n, y, z) = 0,\ n \in \mathbb{N}$.
In particular, a Malcev algebra $A \in \mathcal{M}A_2$ is a tangent algebra of some smooth left automorphic Moufang loop M. Let us call a variety of smooth Moufang loops with the identity $([...[x_1, x_2], x_3]..., x_k], y, z) = 1$ the variety of k-generalized left automorphic Moufang loops.
The tangent algebra of smooth k-generalized left automorphic Moufang loops is an algebra from the variety $\mathcal{M}A_k$. For any algebra from $\mathcal{M}A_k$ there exists a local smooth k-generalized left automorphic Moufang loop.

Theorem

A local smooth k- generalized left automorphic Moufang loop defines a global smooth k-generalized left automorphic Moufang loop.

Global Moufang loops. Literatura

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Grishkov, A.; Rasskazova, M.; LS; Salim, M. *On Malcev algebras nilpotent by Lie center and corresponding analytic Moufang loops*. J. Algebra 575 (2021)

The Theorem of R.Moufang

In 1935 Ruth Moufang proved her famous theorem. Let M be a Moufang loop. If $(a, b, c) = ((ab)c)(a(bc))^{-1} = 1$ for some elements $a, b, c \in M$ then a, b, c generate a subgroup of M. It is easy to see that the Moufang Theorem implies the diassociativity of loops: to verify this fact, it is enough to note that the identity $(a, b, 1) = 1$ holds in all loops.

At the conference "LOOPS 11" in Trest, Czech Republic, Andrew Rajah proposed the following question:
"We say that a variety V of loops satisfies the Moufang theorem if for every loop Q in V the following implication holds: for every $x, y, z \in Q$ if $x(yz) = (xy)z$ then the subloop generated by x, y, z is a group. Is every variety that satisfies the Moufang theorem contained in the variety of Moufang loops?"

Moufang Theorem for Malcev algebras

The analog of the Moufang Theorem in the context of Malcev algebras has the following form. Let $\mathfrak{M}$ be a Malcev algebra. If for given three elements $\{x_1, x_2, x_3\}$ of $\mathfrak{M}$ the equality $J(x_1, x_2, x_3) = 0$ is satisfied, then the subalgebra of $\mathfrak{M}$ generated by these elements, $\{x_1, x_2, x_3\}$, is a Lie algebra.

Thus, Rajah's question in this sense should be: does there exist a variety of binary-Lie algebras satisfying the analog of the Moufang theorem which does not belong to the variety of Malcev algebras?

Moufang Theorem 1

Theorem

Let $\mathfrak{B}$ be a variety of binary Lie algebras defined by the identities

$$x^2 = 0, \quad J(x, y, zu) = 0,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$.
Then
1. $\mathfrak{B}$ is not a variety of Malcev algebras.
2. Every algebra from $\mathfrak{B}$ satisfies the analog of the Moufang Theorem.

Moufang Theorem 2

Proof of the statement 1.
Consider a non-nilpotent solvable 4-dimensional algebra $\mathcal{L}$ from $\mathfrak{B}$ generated by the elements $\{a, b, c\}$ with the following relations:

$$ab = ac = 0, \quad bc = d, \quad da = d, \quad bd = cd = 0.$$

The multiplication is anticommutative. Hence we have $J(a, b, c) = d$, and therefore by direct computation we get: $J(x, y, zu) = 0$,

$$J(\mathcal{L}) = \mathcal{L}^2 = \mathbb{R}d = Lie(\mathcal{L}),$$

where $J(\mathcal{L})$ is an ideal generated by all *jacobians* $J(x, y, z)$ for $x, y, z \in \mathcal{L}$. $Lie(\mathcal{L}) = \{x \in \mathcal{L} \mid J(x, y, z) = 0, \forall y, z \in \mathcal{L}\}$ is the *Lie center* of an algebra $\mathcal{L}$.
We have $J(a, b, ac) - J(a, b, c)a = -d \neq 0$, hence $\mathcal{L}$ is not a Malcev algebra.

Moufang Theorem 3

By the "nilpotent loop" we will understand the concept of commutative nilpotency for loops, i.e. the nilpotency to respect to commutators of a loop.
Let $\mathfrak{B}_n \subset \mathfrak{B}$ be a variety of nilpotent algebras of class n. By B_n let us denote the variety of diassociative loops of nilpotency of the class n satisfying the identity

$$(x, y, [z, t]) = (x, [z, t], y) = ([z, t], x, y) = 1.$$

Using the Malcev Theorem on the correspondence between the analytic diassociative loops and their tangent binary Lie algebras we get the following results.

Moufang Theorem 4

Theorem

Let G be an analytic diassociative loop from the variety B_n. Then the statement of Moufang Theorem holds for G and the corresponding binary-Lie algebra $L(G)$ is contained in the variety $\mathfrak{B}_n$.
Conversely, if L is a finite-dimensional $\mathbb{R}$-algebra from the variety $\mathfrak{B}_n$, then there exists an analytic loop from the variety B_n such that $L(G) \simeq L$.

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Moufang Theorem. Literatura.

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Levi -Malcev Theorem 1

Definition A subalgebra T of the algebra L is called toral if

$$L = \bigoplus_{\alpha \in \Delta} L_\alpha,$$

where for $\alpha \in Hom_{\mathbf{F}}(T, \mathbf{F})$, $L_\alpha = \{x \in L | hx = \alpha(h)x, \text{ for all } h \in T\}$ and Δ is the set of functionals α for which $L_\alpha \neq \{0\}$.
It is clear that $[T, T] = 0$.
Definition A finite-dimensional BL -algebra L is called split, if

$$L = P \oplus T \oplus N,$$

where P is a semisimple Levi factor, T is a toral subalgebra, $[T, P] = 0$, and N is a nilpotent ideal of L.

Levi - Malcev Theorem 2

Theorem. Let L be a finite dimentional algebra from the variety $\mathfrak{B}$ over an algebraically closed field of characteristic 0.
Then $L = S \oplus R(L)$, where $R(L)$ is a solvable radical of L and S is a semisimple Lie algebra. Moreover, L may be embedded in some split algebra $\tilde{L}$ from $\mathfrak{B}$, such that $L^2 = \tilde{L}^2$.

Levi -Malcev Theorem. Literatura

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Moens Theorem 1

Now we will describe some results concerning so-called Leibniz derivations. The concept of *Leibniz derivation* first appeared in the article of W.Moens, and later more generally was studied en the article of Kaygorodov y Popov.. Following we give

1. W. Moens, *A characterisation of nilpotent Lie algebras by invertible Leibniz-derivations,*
Comm. Algebra 41(7) (2013) 2427–2440
2. Kaygorodov, I; Popov, Yu. *A characterization of nilpotent nonassociative algebras by invertible Leibniz-derivations.* J. Algebra 456 (2016), 323–347.

Moens Theorem 1

Definition
Let A be an algebra. $n \in \mathbb{N}, n \geq 2$ and f be an arrangement of brackets on a product of length n. A linear mapping $d : A \to A$ is called an $f-$Leibniz derivation of A if for any $a_1, ..., a_n \in A$ one has:

$$d([a_1, ..., a_n]_f) = \sum_{i=1}^{n} [a_1, a_2, ..., d(a_i), ..., a_n]_f$$

In particular, *left*- Leibniz derivation corresponds to the left arrangements of brackets of length n, that is

$$[a_1, ..., a_n]_{l(n)} = ((...(x_1 x_2)...)x_{n-1})x_n$$

Analogously the *right* Leibniz derivation of length n is defined. For anticommutative algebras the notions of a left Leibniz derivation and a right Leibniz derivation coincide. Moens considered a right Leibniz derivation of a Lie algebra.

Moens Theorem 2

Theorem.(Moens) A finite- dimensional Lie algebra over a field of characteristic zero is nilpotent if and only if it has invertible Leibniz derivation.

Kaygorodov and Popov proved the analogous result for finite -dimensional Malcev algebra over a field of characteristic zero. Authors used the existence of Levi decomposition for Lie algebra and for a Malcev algebra. In the second article the

Open Problem arises: Does the Moens theorem hold in the class of Binary Lie algebras?

Moens Theorem 3

The existence of Levi decomposition for the Binary Lie algebras from the variety $\mathfrak{B}$ was shown above. The structure of algebras from $\mathfrak{B}$ leads to the following

Conjecture Moens Theorem holds for the Binary Lie algebras with the identity $J(x, y, zt) = 0$.

Thank you