Charles University Prague - Department of Algebra

On Conjugation Quandle Coloring of Torus Knots: a Characterization of $GL(2,q)$-Colorability

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Outline

1. Fundamentals of Knot Theory

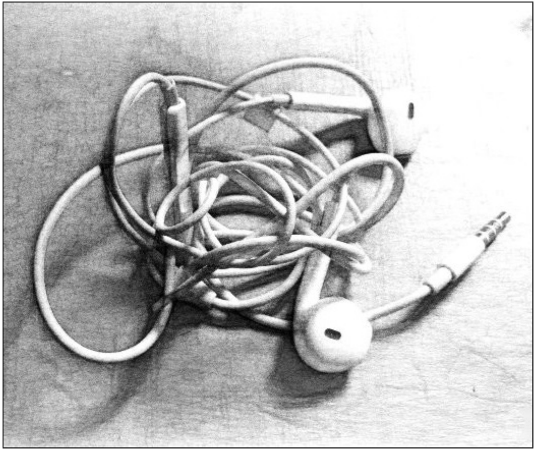
2. Torus Knots and Quandles

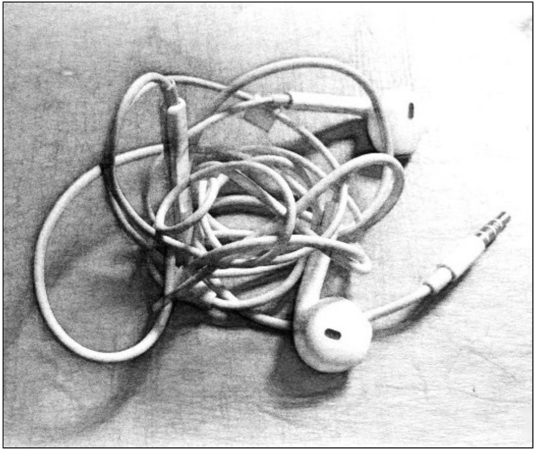
3. Coloring with matrices

F. Spaggiari, *On conjugation quandle coloring of torus knots: a characterization of $\mathsf{GL}(2, q)$-colorability*, Work in progress, **2023**

1. Fundamentals of Knot Theory

What is a knot?





This is not a *mathematical* knot!

What is a knot, formally?

Having loose ends oversimplifies the situation. We need to *glue the ends*.

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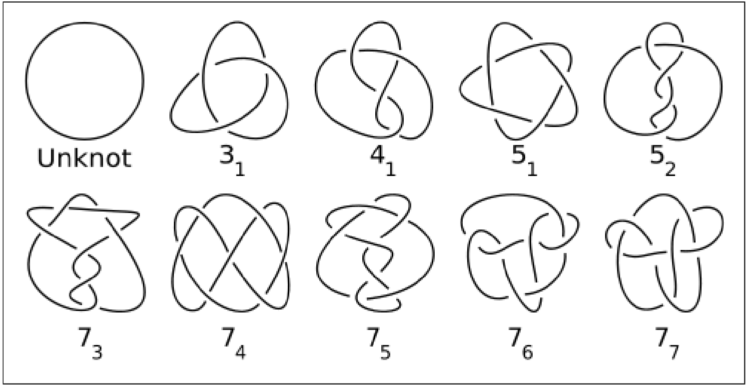
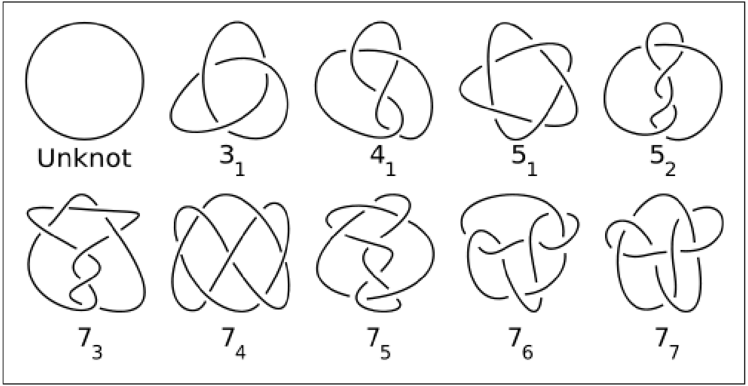
Equivalence Problem: determine if two given knots can be continuously deformed one into the other, aiming the *classification*.

Classification of knots





Remark: K can be untangled $\iff$ K is equivalent to the unknot.

Classification techniques

Definition (Knot invariant)

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Our invariant is **coloring**: we associate a mathematical object with every **strand** of the knot such that at each **crossing** some conditions are fulfilled.

Where is the Algebra behind knots...?

Definition (Quandle)

A **quandle** is a binar $(Q, \triangleright)$ such that for all $x, y, z \in Q$

1. Idempotency: $x \triangleright x = x$

2. Right self-distributivity: $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

3. Right invertibility: $w \triangleright x = y$ has a unique solution $w \in Q$.

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Example (Conjugation quandle)

Let G be a group and define $x \triangleright y = yxy^{-1}$. Then $(G, \triangleright)$ is a *conjugation quandle*, denoted by $\mathsf{Conj}(G)$.

Remark: Of particular interest is $\mathsf{Conj}(\mathsf{GL}(2, q))$: it produces satisfactory results while being reasonably handy.

Quandles are bizarre

Proposition

Let $(Q, \triangleright)$ *be a quandle.*

1. $\triangleright$ *is associative* $\implies$ $(Q, \triangleright)$ *is a trivial quandle.*
2. $\triangleright$ *has an identity element* $\implies$ $(Q, \triangleright)$ *is a trivial quandle.*

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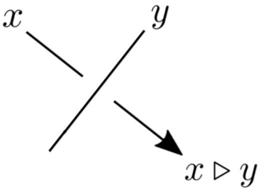
Quandles can be used for coloring knots!

Definition (Quandle coloring)

A $(Q, \triangleright)$**-coloring** of a knot K is a way to associate elements of Q with the strands of K such that at every crossing of K

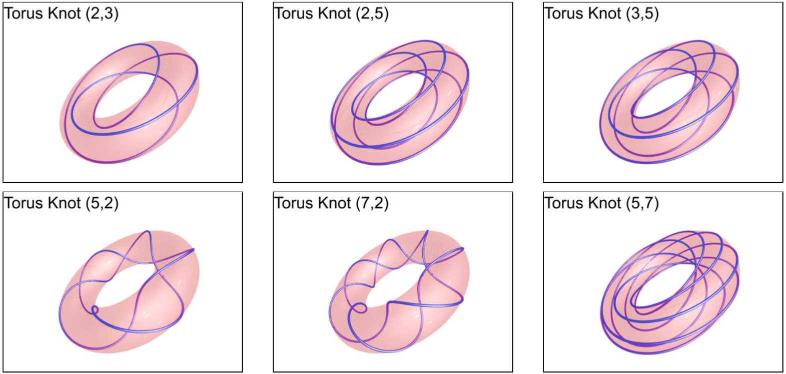
$$x \text{ under } y \text{ produces } z \text{ in } K \quad \Longleftrightarrow \quad x \triangleright y = z \text{ in } (Q, \triangleright).$$



Only **non-trivial colorings** are interesting.

Definition (Torus Knot)

A **torus knot** is any knot that can be embedded on the trivial torus.



Insight on torus knots

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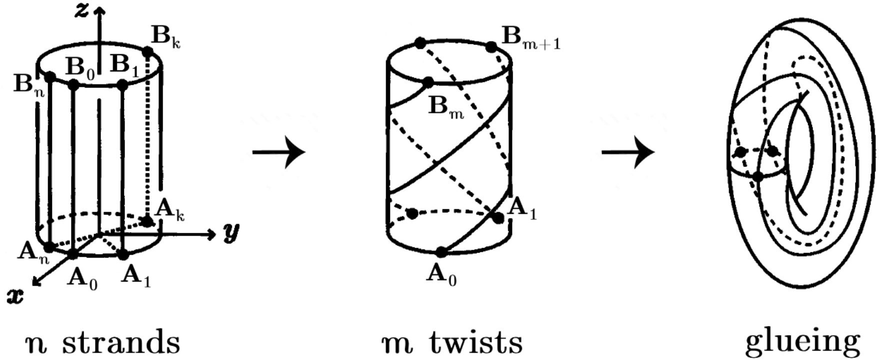
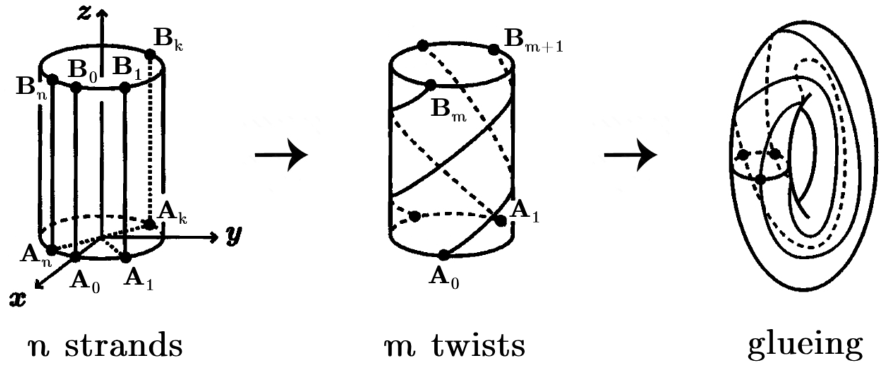
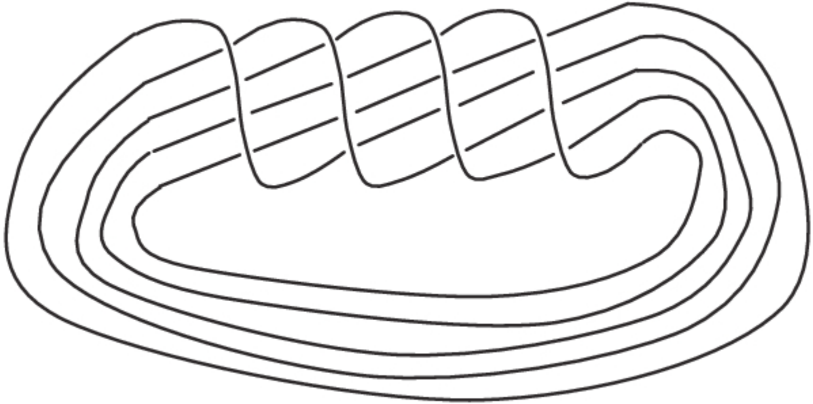

z
x
y
B_k
B_0
B_1
B_n
A_k
A_n
A_0
A_1
B_{m+1}
B_m
A_1
A_0
n strands
m twists
glueing

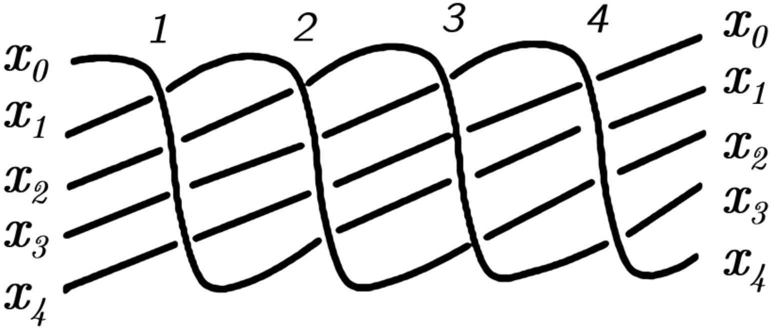


Notation K(m, n)

The **torus knot** with n strands and m twists will be denoted by K(m, n).

2D diagram representation





$\mathsf{K}(4,5)$

2. Torus Knots and Quandles

<u>**Problem:**</u>

$\mathsf{K}(m,n)$ **is** $\mathsf{Conj}(G)$**-colorable**

$$\Updownarrow$$

some conditions in G **hold**

Conjugation quandle coloring of $\mathsf{K}(m, n)$

Theorem

Let G be a group. The following are equivalent:

1. $\mathsf{K}(m, n)$ *is* $\mathsf{Conj}(G)$*-colorable.*
2. $\exists x_0, \ldots, x_{n-1} \in G$ *such that all the following terms are equal*

$$\{x_{\sigma^k(0)} x_{\sigma^k(1)} \ldots x_{\sigma^k(m-1)} : k = 0, \ldots, n-1\},$$

   where $\sigma = (0\ 1\ 2\ \ldots\ n-1) \in \mathsf{S}_n$ *is a cyclic permutation of the indices.*

3. $\exists x_0, \ldots, x_{n-1} \in G$ *such that for* $u = x_{n-m} x_{n-m+1} \ldots x_{n-2} x_{n-1}$ *we have*

$$x_i \triangleright u = x_{i-m} \pmod{n} \qquad \forall i = 0, \ldots, n-1.$$

Remark: It translates a geometric coloring condition only in terms of quandle or group equations (*n.b.* quandles are nice, but groups are better!).

Inducing colorings

Proposition

Let $y_0, \ldots, y_{tn-1}$ *be a coloring of* $\mathsf{K}(m, tn)$. *Define*

$$x_i = \prod_{j=0}^{t-1} y_{it+j}, \qquad \text{for } i = 0, \ldots, n-1.$$

Then $x_0, \ldots, x_{n-1}$ *is a (possibly trivial) coloring of* $\mathsf{K}(m, n)$.

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Corollary

Let $x_0, \dots, x_{p-1}$ be a coloring of $\mathsf{K}(m, p)$. Then either it is the trivial coloring or all the colors are distinct.

Weakening the problem

Theorem

$\mathsf{K}(m,n)$ is $\mathsf{Conj}(G)$-colorable if and only if there is a prime factor p of m and a prime factor q of n such that $\mathsf{K}(p,q)$ is $\mathsf{Conj}(G)$-colorable.

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Theorem

Let $m \in \mathbb{N}$ and p be a prime such that $p \nmid m$. Then $\mathsf{K}(m,p)$ is $\mathsf{Conj}(G)$-colorable if and only if there is $u \in G$ such that the centralizers $\mathsf{C}_G(u^p) \setminus \mathsf{C}_G(u) \neq \emptyset$.

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Remark: The colorability of $\mathsf{K}(m,p)$

- Depends on a single element $u \in G$.
- It does not depend on m.

3. Coloring with matrices

Problem:

$\mathsf{K}(m, p)$ **is** $\mathsf{Conj}(\mathsf{GL}(2, q))$**-colorable**

$$\Updownarrow$$

$f(m, p, q)$ **holds**

We know the conjugacy classes of G, the representatives, and their centralizers.

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Type	u	$\mathsf{C}_{\mathsf{GL}(2,q)}(u)$
Type 1	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\mathsf{GL}(2,q)$
Type 2	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	$\left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathsf{GL}(2,q) \colon u, v \neq 0 \right\}$
Type 3	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\left\{ \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in \mathsf{GL}(2,q) \colon u \neq 0 \right\}$
Type 4	$\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$	$\left\{ \begin{pmatrix} u & v \\ au & u + bv \end{pmatrix} \in \mathsf{GL}(2,q) \colon u \neq 0 \text{ or } v \neq 0 \right\}$

So, when does the centralizer expand?

Type	u^p	$\mathsf{C}_{\mathsf{GL}(2,q)}(u^p) \setminus \mathsf{C}_{\mathsf{GL}(2,q)}(u) \neq \emptyset$
Type 1	$\begin{pmatrix} a^p & 0 \\ 0 & a^p \end{pmatrix}$	Never
Type 2	$\begin{pmatrix} a^p & 0 \\ 0 & b^p \end{pmatrix}$	$p \mid q-1$
Type 3	$\begin{pmatrix} a^p & pa^{p-1} \\ 0 & a^p \end{pmatrix}$	$p = q$
Type 4	$\begin{pmatrix} x_{p-1} & y_{p-1} \\ ay_{p-1} & x_{p-1} + by_{p-1} \end{pmatrix}$	$p \mid q+1$

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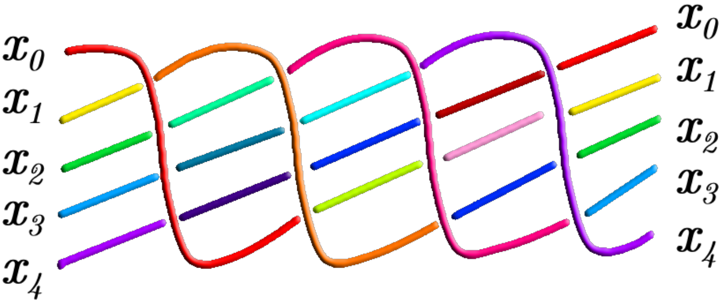
$$\text{where } \begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases} \qquad \begin{cases} x_n = ay_{n-1} \\ y_n = x_{n-1} + by_{n-1}. \end{cases} \qquad n \geq 1.$$

The solution

Theorem (Main Result)

The following conditions are equivalent.

1. $p \mid q(q+1)(q-1)$.



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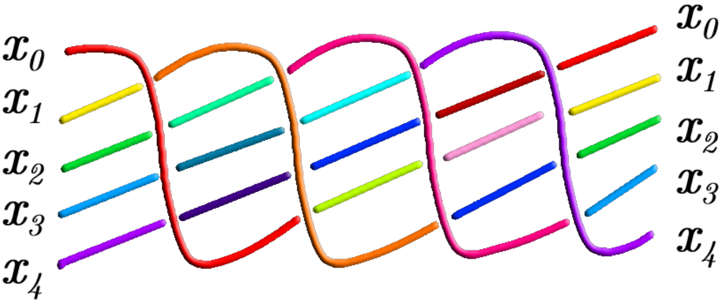

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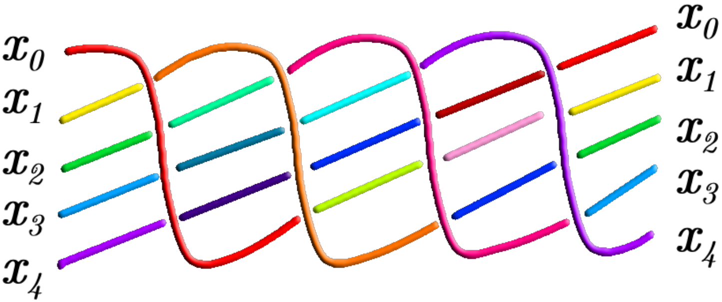
2. $\mathsf{K}(m,p)$ *is* $\mathsf{Conj}(\mathsf{GL}(2,q))$*-colorable.*
3. $\mathsf{K}(m,p)$ *is* $\mathsf{Conj}(\mathsf{SL}(2,q))$*-colorable.*



Conclusions

Summary:

- We have developed tools to analyze $\mathsf{Conj}(G)$-coloring of a torus knot $\mathsf{K}(m,n)$.
  - We may assume m,n to be primes.
  - The colorability only depends on n and on one element in the group.
- Taking $G = \mathsf{GL}(2,q)$ or $G = \mathsf{SL}(2,q)$, we have completely characterized the colorability in terms of a numeric condition involving divisibility.

New horizons:

- $\mathsf{Conj}(G)$-coloring of $\mathsf{K}(m,p)$ for other groups G.
- Relations among $\mathsf{Conj}(G)$-coloring and the Jones polynomial.
- $\mathsf{Conj}(G)$-coloring of the Whitehead double of $\mathsf{K}(m,p)$.

That’s all, thanks!

Bibliography I

[1] F. Spaggiari, *On conjugation quandle coloring of torus knots: a characterization of* $\mathsf{GL}(2,q)$*-colorability*, Work in progress, **2023**.

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[3] M. Richling, Torus Knots, **2022**, https://www.mitchr.me/SS/torusKnots/index.html#orgcfdc49b (visited on 06/23/2023).

Do you have questions, or knot?

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