# Quantum Latin Squares and Many-Body Quantum Systems

Jamie Vicary
Department of Computer Science
University of Cambridge

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29 June 2023

# Quantum Structures Tour

**Hadamard matrix**

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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Unitary matrix where every entry has the same modulus.

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**Latin square**

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

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**Quantum Latin square**

$$L = \begin{pmatrix} |0\rangle & |1\rangle & |2\rangle & |3\rangle \\ |1\rangle & |0\rangle & |3\rangle & |2\rangle \\ |2\rangle & |3\rangle & |0\rangle & |1\rangle \\ |3\rangle & |2\rangle & |1\rangle & |0\rangle \end{pmatrix}$$

Grid of elements of $\mathbb{C}^n$ where every row and column is a complete orthonormal basis.

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## Unitary basis

$$U = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \text{-}i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \text{-}1 \end{pmatrix} \right)$$

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## Quantum cross

$$C = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ \begin{pmatrix} i & j \\ k & l \end{pmatrix} & \begin{pmatrix} m & n \\ o & p \end{pmatrix} \end{pmatrix}$$

Grid of unitary operators, which remain unitary after an inversion process.

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# Building Quantum Structures

**Hadamard** $H_{i,j}$ | **Quantum Latin square** $L_{i,j,k}$ | **Unitary basis** $U_{i,j,k}$ | **Cross** $C_{i,j,k,l}$

After decades of work, many rich connections are known between these structures.
(These are due to Werner, Hosoya, Suzuki, Dita, Musto, Reutter, Vicary)

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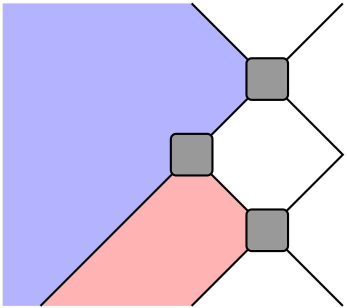
**THESE ARE BEAUTIFUL – BUT MYSTERIOUS!**

# Shaded tensor networks

A *shaded tensor network* is a planar string diagram, where some of the regions are shaded.



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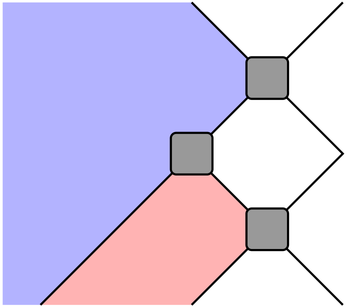

It is a string diagram for a subcategory of the **2Hilb**, the 2-category of 2–Hilbert spaces.

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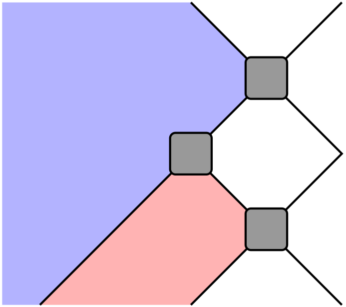

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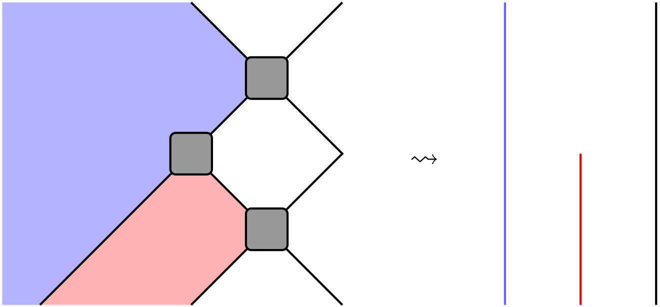

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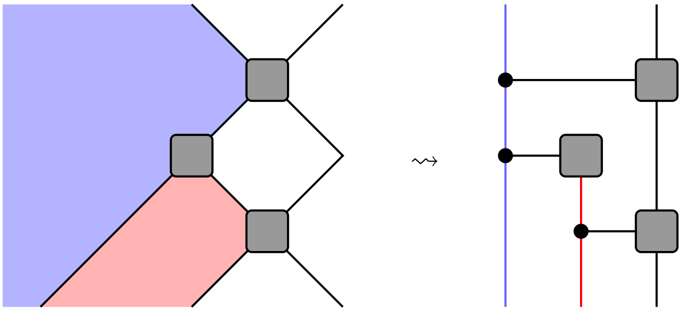

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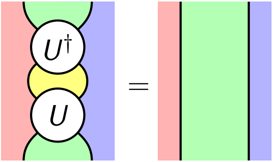
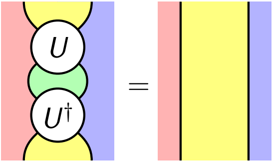
# Biunitarity

In the world of shaded tensor networks, a map $U$ can be vertically unitary:



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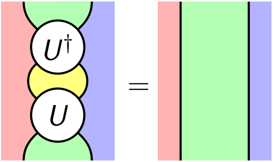
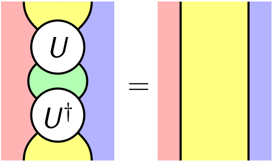
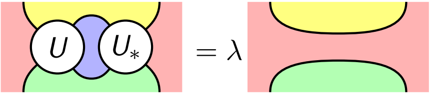
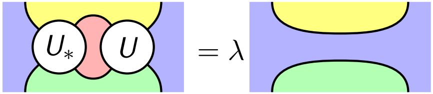



If $U$ is 4-valent, we can also ask if it is *horizontally unitary*:



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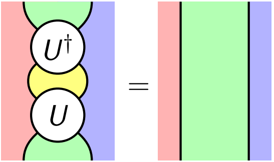


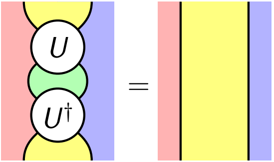


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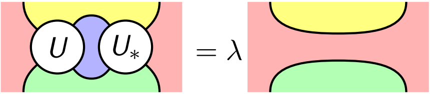


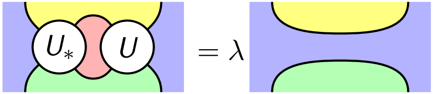


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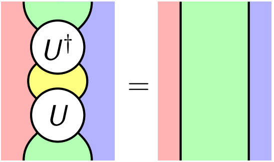


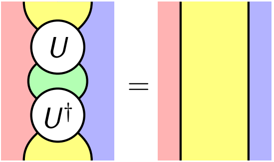


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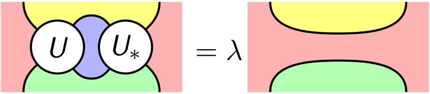


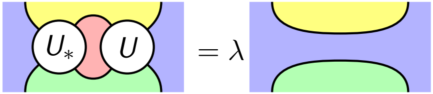


A 4-valent map is *biunitary* when it is vertically and horizontally unitary.

First introduced by Ocneanu in 1989, to study subfactors of von Neumann algebras.

# Quantum structures from biunitaries

We will consider biunitary vertices with different shading patterns.

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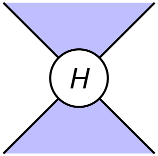

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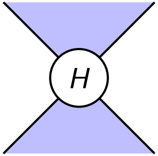


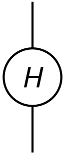

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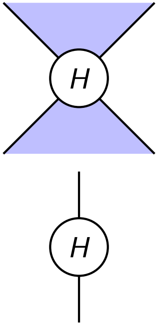


Hadamard

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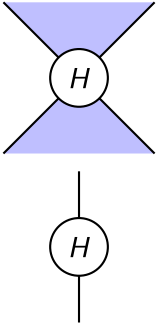


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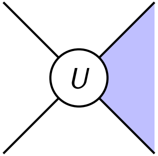

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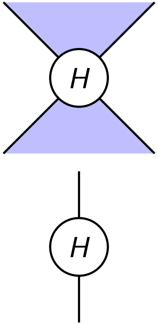


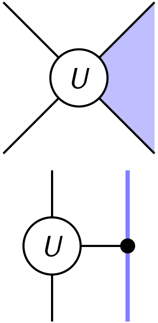


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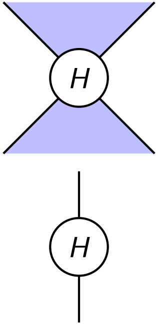


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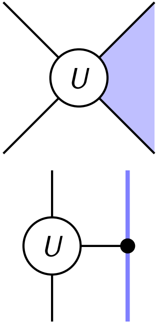


Unitary basis

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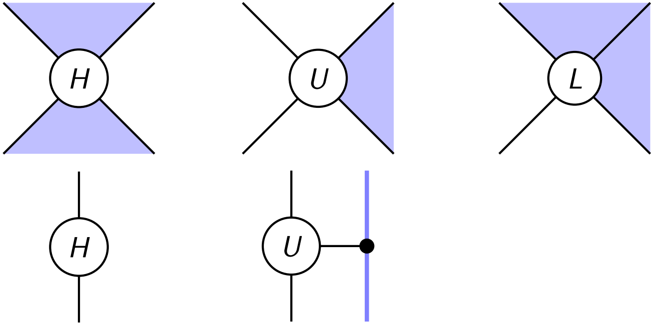


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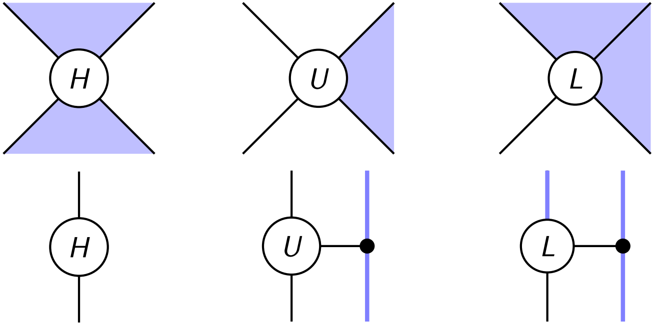


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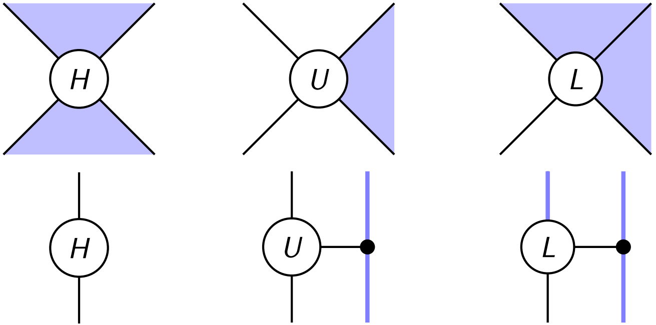


Hadamard | Unitary basis | Quantum Latin square

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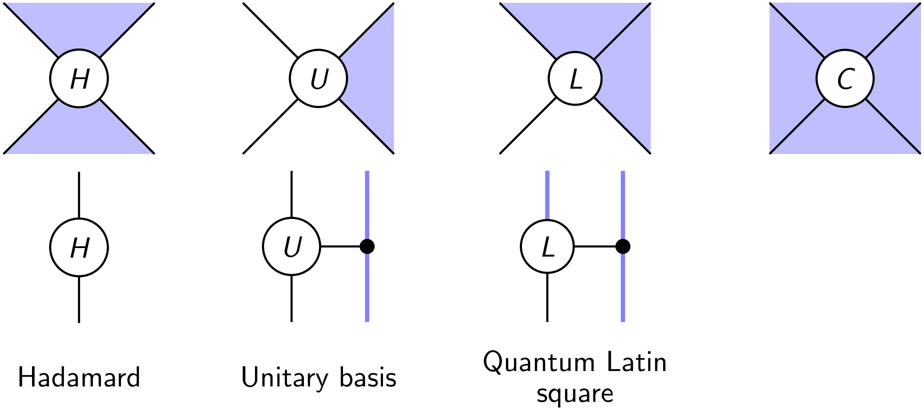

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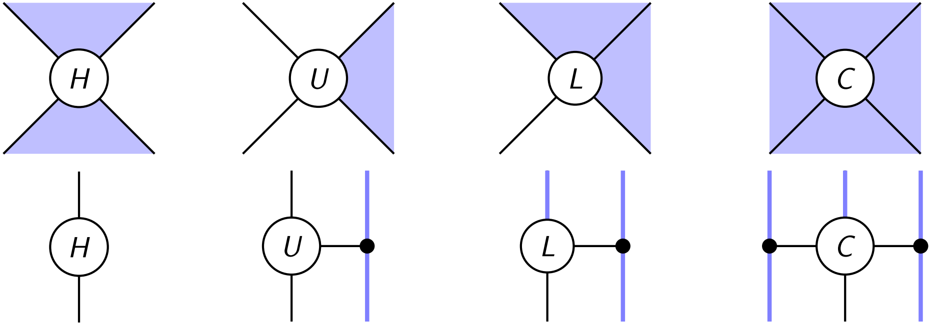


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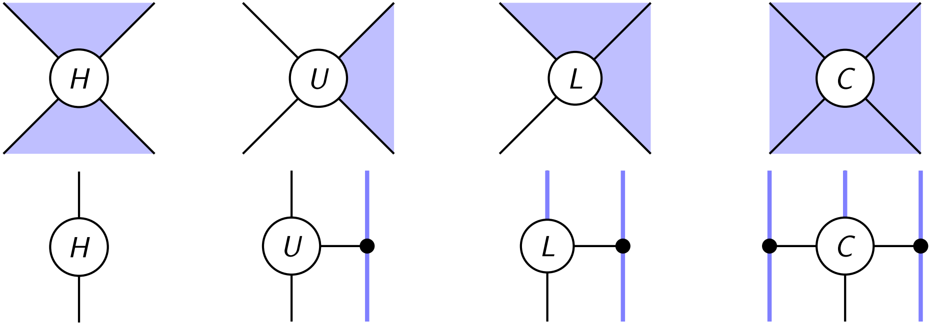


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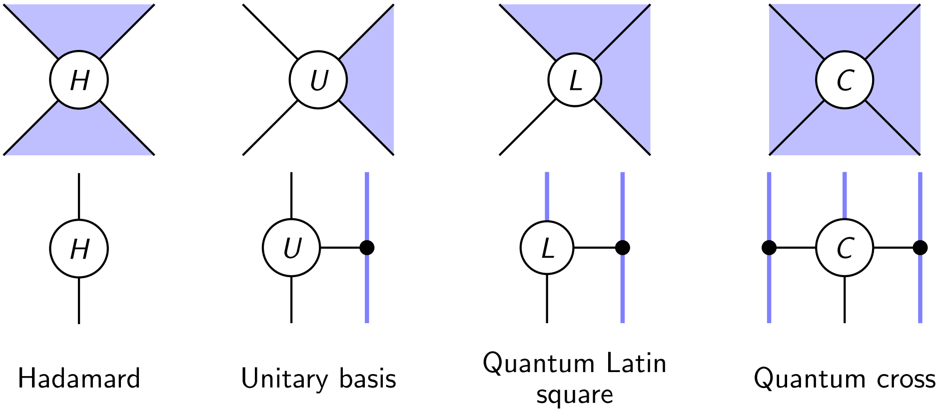


These shading patterns exactly recover our different quantum structures.

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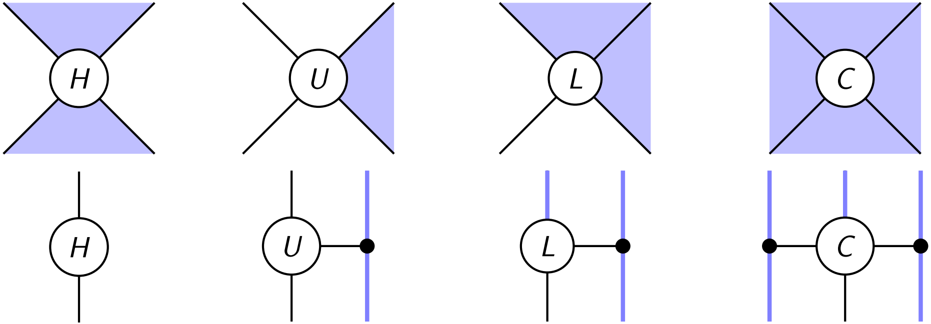


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The Hadamard shading pattern was first discovered by Vaughan Jones in 1989.

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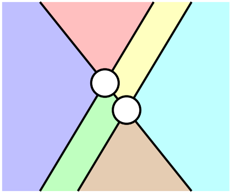

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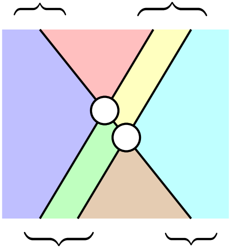

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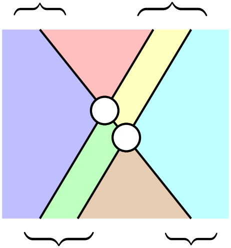

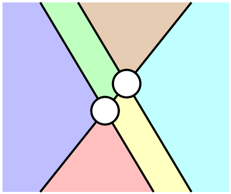

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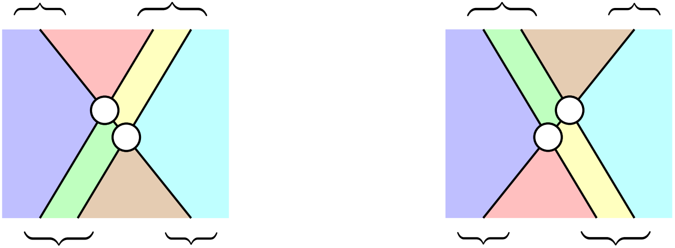

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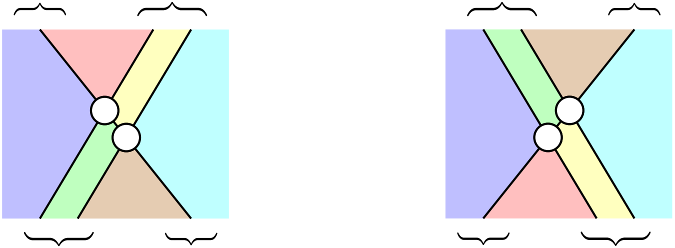

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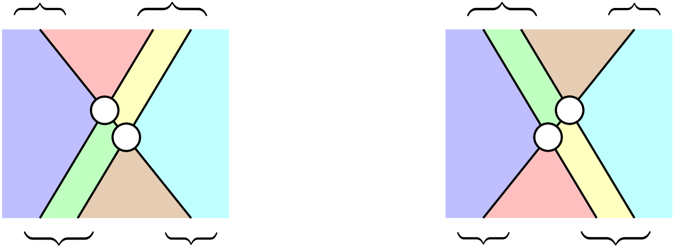

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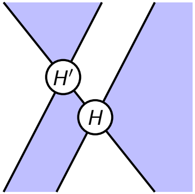
But diagonal composition *changes the shading pattern*!

# Geometrical constructions

By composing our quantum structures diagonally, we can make new ones.



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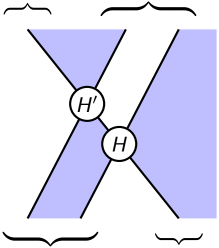

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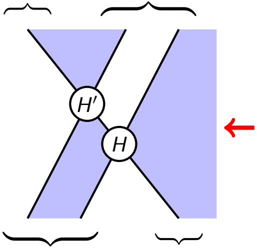

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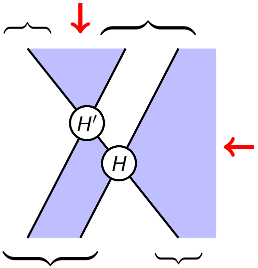

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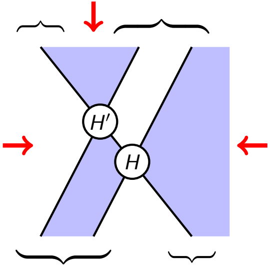

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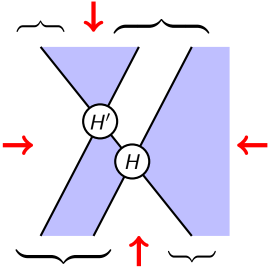

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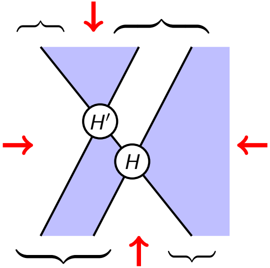


$H + H' \rightsquigarrow L$

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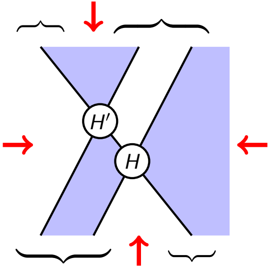


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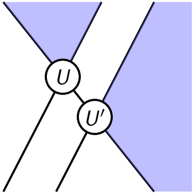

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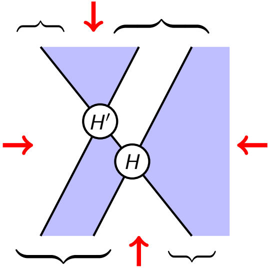


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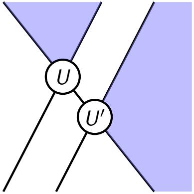


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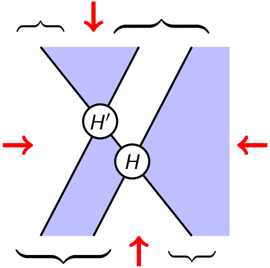


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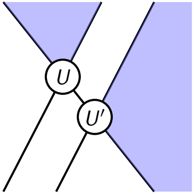


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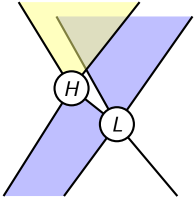

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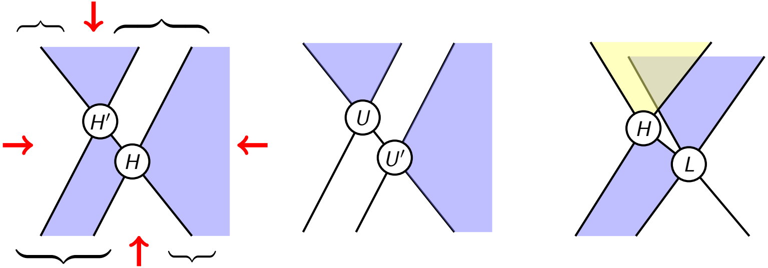


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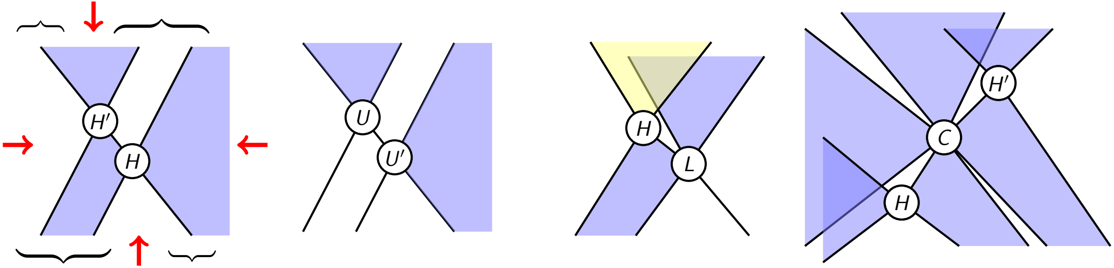


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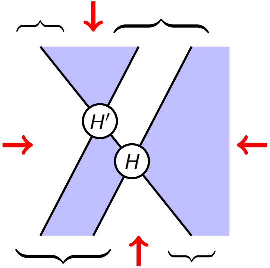


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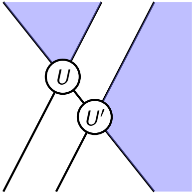


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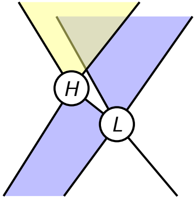


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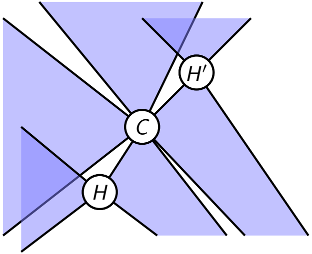


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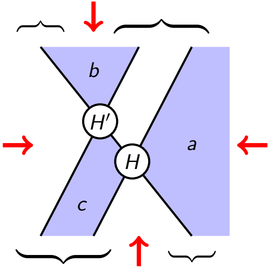


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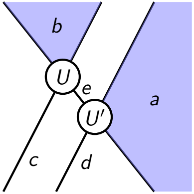


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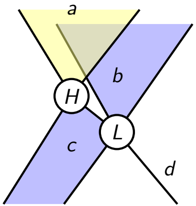


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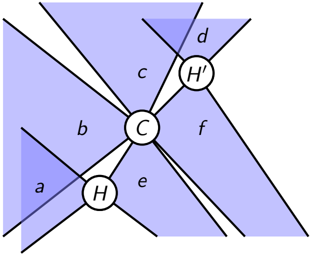


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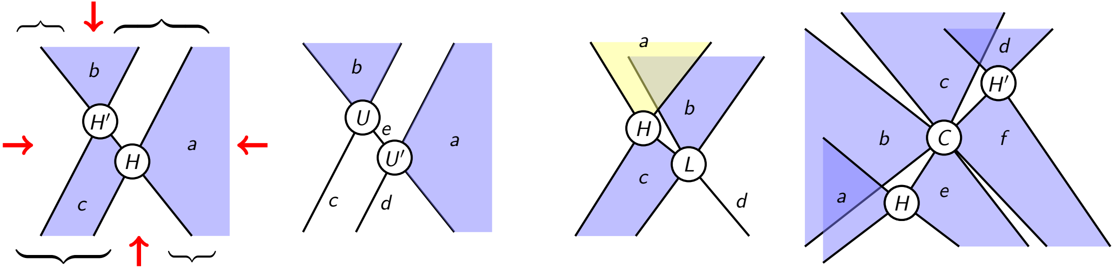


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This shows the true geometrical nature of these constructions.

# 2–Hilbert spaces

$$\begin{matrix} & 1 & \sqrt{2}-i \\ 0 & & \\ & i & \ddots \end{matrix}$$

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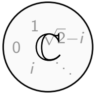

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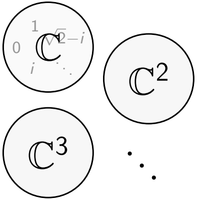

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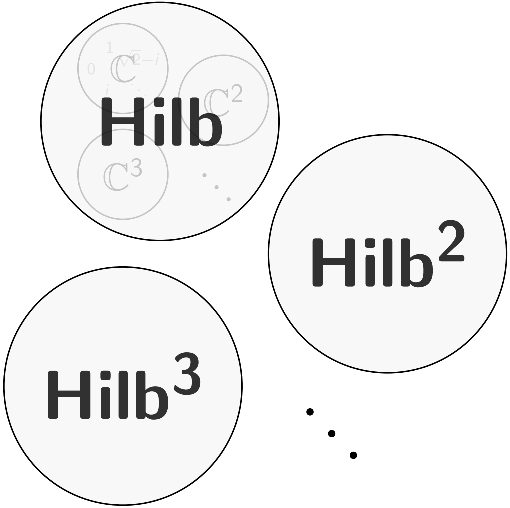

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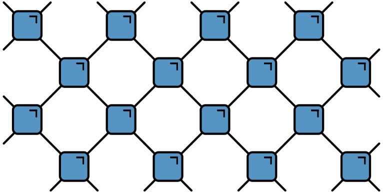

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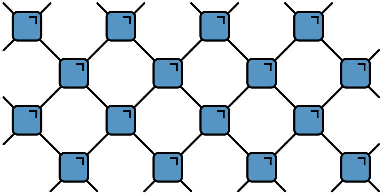
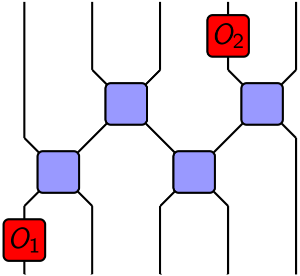


We want to analyze emergent properties of these circuits, and relate them to real systems.

# Correlations outside the light cone

Let's verify this property: trivial correlations for measurements outside the light cone.



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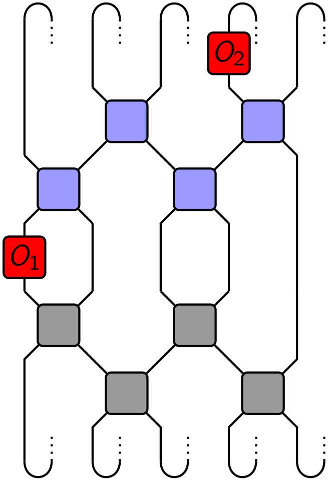

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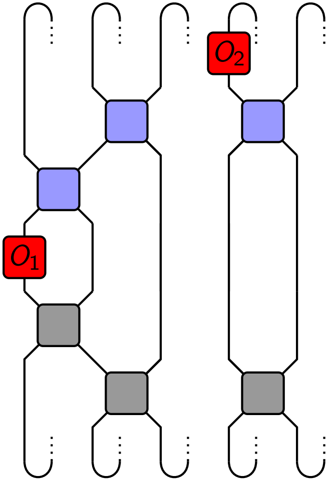

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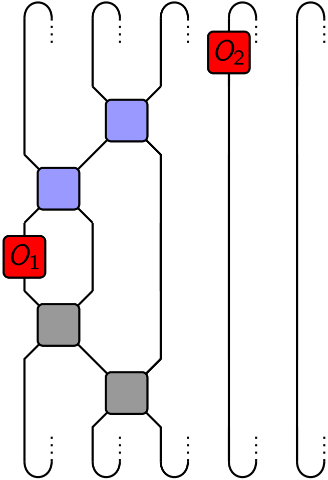

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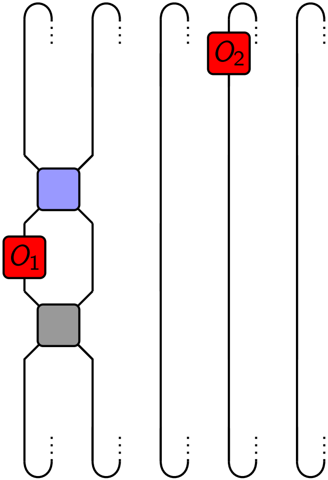

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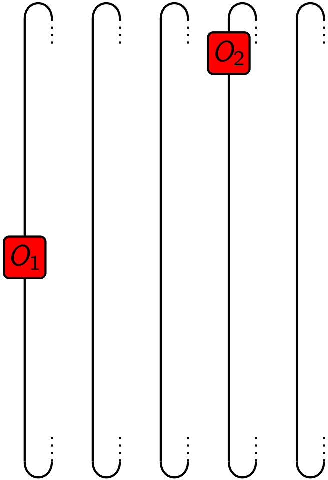

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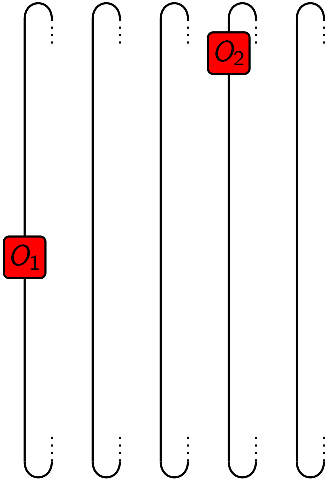
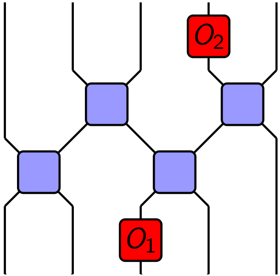



This factorizes, so the correlation is trivial.

# Correlation inside light cone

For measurements *inside* the light cone, the correlations can take any value.



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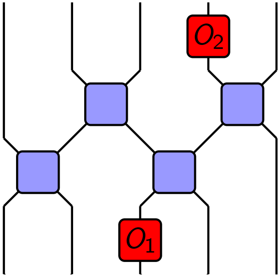


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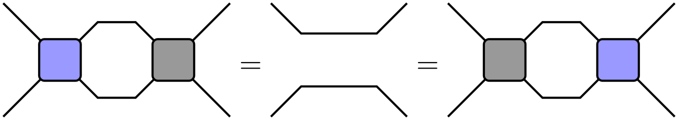

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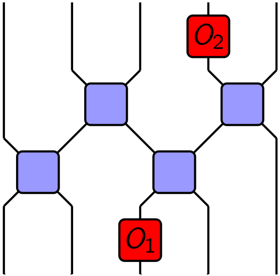


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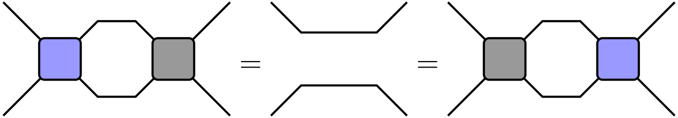


Then applying a space-time symmetry, the previous proof applies.

So correlations inside the light cone are now also trivial!

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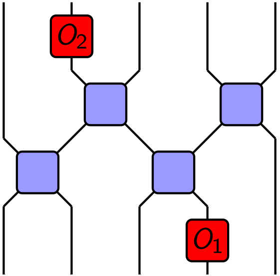


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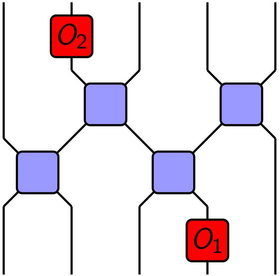


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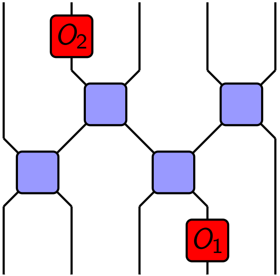


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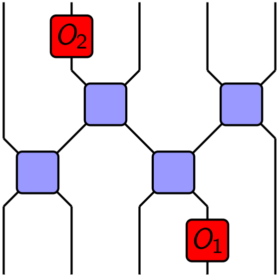


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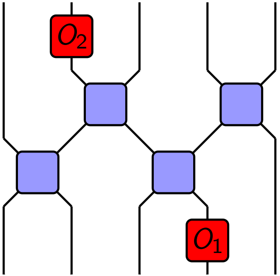


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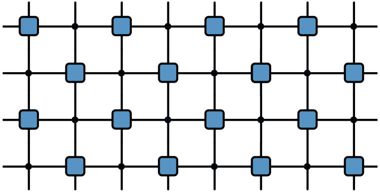

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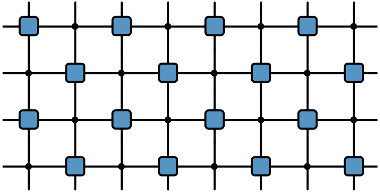

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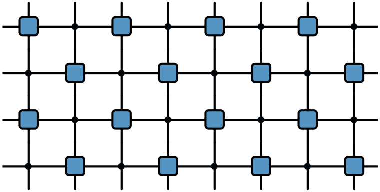

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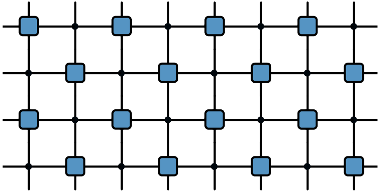

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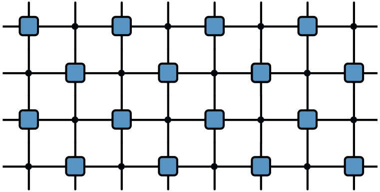

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This is surprising — their structure is very different. **How can we understand this?**

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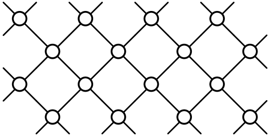

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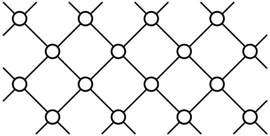

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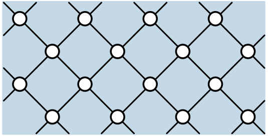

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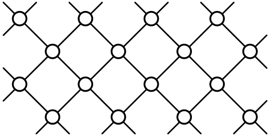

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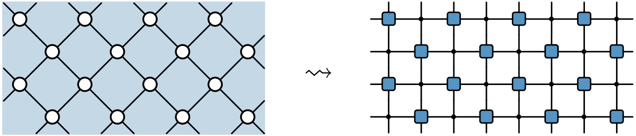

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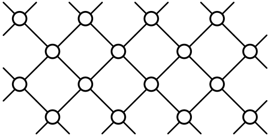

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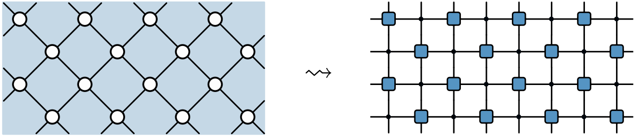

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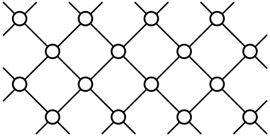

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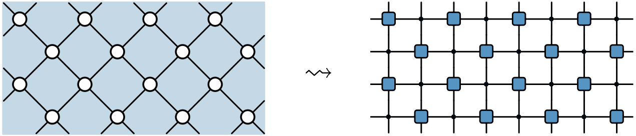

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This also recovers Prosen's definition of dual unitarity for clockwork circuits.

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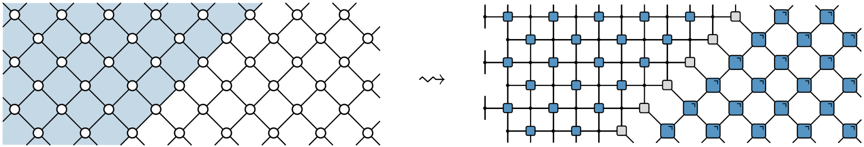

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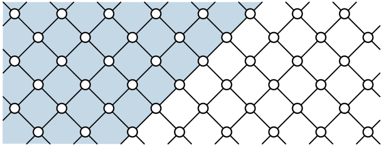
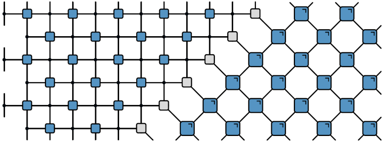


$\rightsquigarrow$



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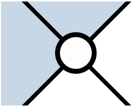
At the boundary we require a new sort of vertex, with two shaded and two unshaded regions.



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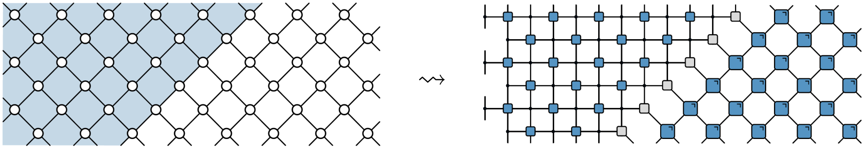

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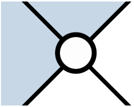

We saw these before — they are quantum Latin squares!

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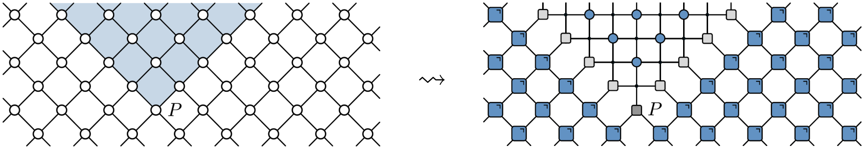


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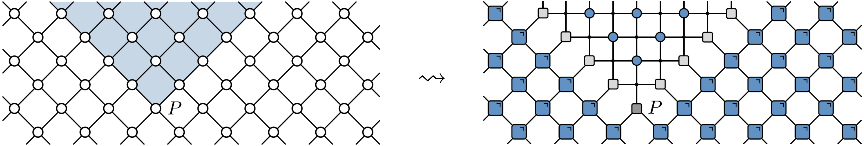


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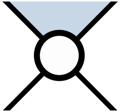
At point $P$ we encounter another new vertex type, with one shaded region.



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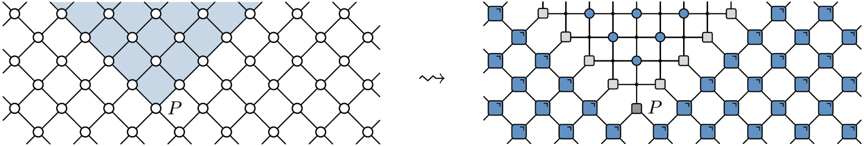
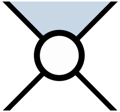



Here a new clockwork region is created within an existing brickwork region.

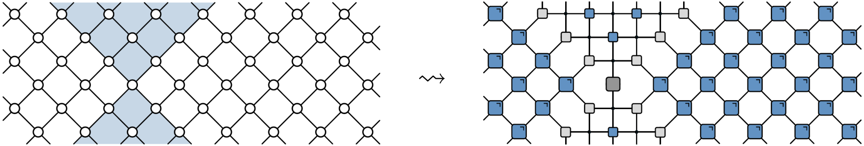
At point $P$ we encounter another new vertex type, with one shaded region.



These are unitary error bases, orthogonal and complete families of unitary matrices.

# Biunitary circuits — boundary reflection

Boundaries can also reflect off each other:



Here a clockwork region contracts to zero width, then expands again.

# Biunitary circuits — boundary reflection

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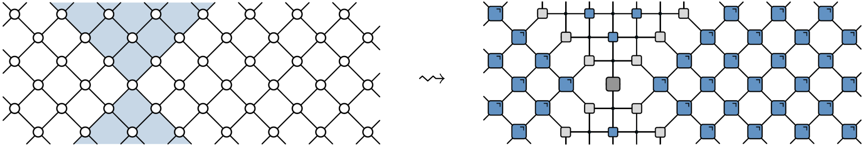
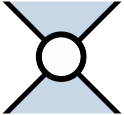


Here a clockwork region contracts to zero width, then expands again.

The central point has a vertex with two non-adjacent shaded regions.



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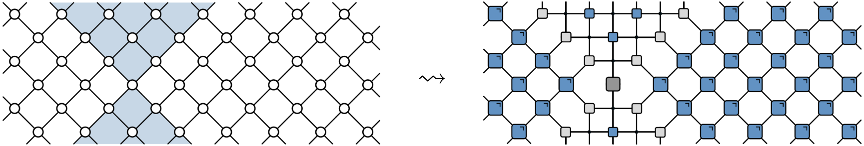

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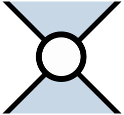
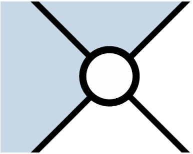


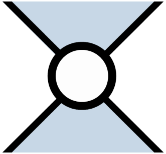
Here we are using Hadamard matrices, unitary matrices where every coefficient has the same absolute value.

# The geometry of quantum structures

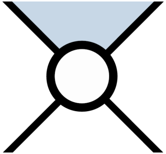
Shaded planar algebra is a tool for exploring the *geometry* of quantum combinatorics.



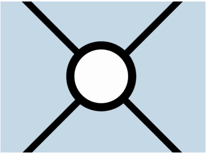
*Quantum Latin Square*



*Hadamard*



*Unitary Basis*



*Quantum Cross*

It will be exciting to use these techniques to discover more about the relationship between quantum physics, quantum combinatorics, and quantum information.