Biracks II

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Outline

Multipermutation biracks of level 2

- 2-reductive
- distributive
- 2-permutational

Equational characterization of biracks

Definition [Stanovský 2006]

An algebraic structure $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ with four binary operations is called a **birack**, if the following holds for any $x, y, z \in X$:

- $x \circ (x \backslash_\circ y) = y = x \backslash_\circ (x \circ y)$
- $(y /_\bullet x) \bullet x = y = (y \bullet x) /_\bullet x$
- $x \circ (y \circ z) = (x \circ y) \circ ((x \bullet y) \circ z)$
- $(x \circ y) \bullet ((x \bullet y) \circ z) = (x \bullet (y \circ z)) \circ (y \bullet z)$
- $(x \bullet y) \bullet z = (x \bullet (y \circ z)) \bullet (y \bullet z)$

Retraction relations of a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$

$$
\begin{aligned}
x \sim y \quad &\Leftrightarrow \quad L_x = L_y \quad \Leftrightarrow \quad \forall a \in X \quad x \circ a = y \circ a \\
x \backsim y \quad &\Leftrightarrow \quad \mathbf{R}_x = \mathbf{R}_y \quad \Leftrightarrow \quad \forall a \in X \quad a \bullet x = a \bullet y \\
x \approx y \quad &\Leftrightarrow \quad L_x = L_y \quad \text{and} \quad \mathbf{R}_x = \mathbf{R}_y
\end{aligned}
$$

The quotient birack $\mathrm{Ret}(X) = (X/_{\approx}, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called the *retraction* of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is of multipermutation level 2 if

$$|\mathrm{Ret}(\mathrm{Ret}(X))| = 1$$

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - a birack

$$|\mathrm{Ret}(\mathrm{Ret}(X))| = 1$$

For every $x, y, z, t \in X$:

$$(z \circ x) \circ y = (t \circ x) \circ y$$
$$(x \bullet z) \circ y = (x \bullet t) \circ y$$
$$y \bullet (z \circ x) = y \bullet (t \circ x)$$
$$y \bullet (x \bullet z) = y \bullet (x \bullet t)$$

Condition $(**)$

Proposition

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *be a birack satisfying the following two properties:*

$$\forall x \in X \; \exists y \in X \quad y \circ x = x,$$
$$\forall x \in X \; \exists y \in X \quad x \bullet y = x.$$

Then $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is a multipermutation birack of level at most* 2 *if and only if it satisfies for* $x, y, z \in X$*:*

$$(x \circ y) \circ z = y \circ z$$
$$z \bullet (y \bullet x) = z \bullet y$$
$$(y \bullet x) \circ z = y \circ z$$
$$z \bullet (x \circ y) = z \bullet y$$

2-reductive biracks

Definition

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is called 2-reductive if it satisfies

$$(x \circ y) \circ z = y \circ z$$
$$z \bullet (y \bullet x) = z \bullet y$$
$$(y \bullet x) \circ z = y \circ z$$
$$z \bullet (x \circ y) = z \bullet y$$

Proposition

If a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-reductive then the permutation group $\mathrm{Mlt}(X)$ is abelian.

A birack:

- $x \circ (y \circ z) = (x \circ y) \circ ((x \bullet y) \circ z)$
- $(x \circ y) \bullet ((x \bullet y) \circ z) = (x \bullet (y \circ z)) \circ (y \bullet z)$
- $(x \bullet y) \bullet z = (x \bullet (y \circ z)) \bullet (y \bullet z)$

2-reductive biracks - construction

Theorem

I - a non-empty set

$(A_i)_{i\in I}$ - a family of abelian groups over I

$\bigcup_{i\in I} A_i$ - the disjoint union of the sets A_i

$c_{i,j}, d_{i,j} \in A_j$, for $i,j \in I$, - some constants

Then $(\bigcup_{i\in I} A_i, \circ, \backslash_\circ, \bullet, /_\bullet)$, where for $x \in A_i$, $y \in A_j$,

$$x \circ y = y + c_{i,j} \quad x\backslash_\circ y = y - c_{i,j}$$
$$x \bullet y = x + d_{j,i} \quad x/_\bullet y = x - d_{j,i}$$

is a 2-reductive birack.

The disjoint union of abelian groups

Remark

The birack $(\bigcup_{i\in I} A_i, \circ, \backslash_\circ, \bullet, /_\bullet)$ *satisfying*

$$A_j = \langle \{c_{i,j}, d_{i,j} \mid i \in I\} \rangle\,, \text{ for every } j \in I,$$

has orbits of the action of $\mathrm{Mlt}(X)$ *equal to* A_j*,* $j \in I$ *and each orbit is a permutational birack.*

We call it the disjoint union, over a set I*, of abelian groups and denote by:*

$$\mathcal{A} = ((A_i)_{i\in I},\, (c_{i,j})_{i,j\in I},\, (d_{i,j})_{i,j\in I}).$$

Theorem

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is 2-reductive if and only if it is a disjoint union, over a set* I*, of abelian groups.*

The orbits of the action of $\mathrm{Mlt}(X)$ *coincide with the groups.*

Isomorphism theorem

Theorem

$\mathcal{A} = ((A_i)_{i\in I},\ (c_{i,j})_{i,j\in I},\ (d_{i,j})_{i,j\in I})$ *and*
$\mathcal{A}' = ((A'_i)_{i\in I},\ (c'_{i,j})_{i,j\in I},\ (d'_{i,j})_{i,j\in I})$

are isomorphic 2*-reductive biracks if and only if there is a bijection* π *of the set* I *and group isomorphisms* $\psi_i\colon A_i \to A'_{\pi(i)}$ *such that*

$$\psi_j(c_{i,j}) = c'_{\pi(i),\pi(j)} \quad \text{and} \quad \psi_j(d_{i,j}) = d'_{\pi(i),\pi(j)},$$

for every $i,j \in I$.

How to construct all biracks of multipermutation level 2 of size n

Algorithm

1. *For all partitionings* $n = n_1 + n_2 + \cdots + n_k$ *do (2)–(4).*
2. *For all abelian groups* $A_1, \ldots, A_k$ *of size* $|A_i| = n_i$ *do (3)–(4).*
3. *For all constants* $c_{i,j}, d_{i,j} \in A_j$*, for all* $1 \leq i,j \leq k$ *do (4).*
4. *If, for all* $1 \leq i,j \leq k$*, we have* $A_j = \langle \{c_{i,j}, d_{i,j} \mid i \in I\} \rangle$ *then construct a birack* $(\bigcup A_i, (c_{i,j}), (d_{i,j}))$*:*

$$x \circ y = y + c_{i,j} \quad x \backslash_\circ y = y - c_{i,j}$$
$$x \bullet y = x + d_{j,i} \quad x /_\bullet y = x - d_{j,i}$$

where $x \in A_i$ *and* $y \in A_j$*.*

The disjoint union of abelian groups

Remark

$\mathcal{A} = ((A_i)_{i\in I},\ (c_{i,j})_{i,j\in I},\ (d_{i,j})_{i,j\in I})$:

- *is idempotent if and only if* $c_{i,i} = d_{i,i} = 0$*, for each* $i \in I$
- *is involutive if and only if* $d_{i,j} = -c_{i,j}$*, for each* $i, j \in I$
- *satisfies Condition* $(**)$ *if and only if*

$$\forall j \in I \quad \exists i, i' \in I, \quad \text{such that} \quad c_{i,j} = d_{i',j} = 0$$

- Idempotent birack: $x \circ x = x$ and $x \bullet x = x$
- Involutive birack: $x \bullet y = (x \circ y)\backslash_{\circ} x$
- Condition $(**)$: $\forall x \in \ \exists y \in X \quad y \circ x = x$ and $\forall x \in X\ \exists y \in X \quad x \bullet y = x$

The retraction relations

Proposition

For 2-reductive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$, *the retraction relations:*

$$x \sim y \quad \Leftrightarrow \quad L_x = L_y \quad \Leftrightarrow \quad \forall a \in X \quad x \circ a = y \circ a$$
$$x \backsim y \quad \Leftrightarrow \quad \mathbf{R}_x = \mathbf{R}_y \quad \Leftrightarrow \quad \forall a \in X \quad a \bullet x = a \bullet y$$
$$x \approx y \quad \Leftrightarrow \quad L_x = L_y \quad \text{and} \quad \mathbf{R}_x = \mathbf{R}_y$$

are congruences.

The quotient biracks: $\mathrm{Ret}(X)$, $\mathrm{LRet}(X)$ *and* $\mathrm{RRet}(X)$ *are projection ones.*

Retraction relations for 2-reductive biracks

$$(X, \circ, \backslash_\circ, \bullet, /_\bullet) = ((A_i)_{i\in I},\ (c_{i,j})_{i,j\in I},\ (d_{i,j})_{i,j\in I})$$

Lemma

For $x \in A_i$ and $y \in A_j$

$$x \sim y \quad \Leftrightarrow \quad \forall (k \in I)\ c_{i,k} = c_{j,k}$$

and

$$x \backsim y \quad \Leftrightarrow \quad \forall (k \in I)\ d_{i,k} = d_{j,k}$$

Distributive biracks $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$

- **left distributive** if $(X, \circ, \backslash_\circ)$ is a left rack
- **right distributive** if $(X, \bullet, \backslash_\bullet)$ is a right rack
- **distributive** if it is left and right distributive

Example

$n \in \mathbb{N}$ - an odd natural number

The birack $(\mathbb{Z}_n, \circ, \backslash_\circ, \bullet, /_\bullet)$, with

$$\begin{aligned} x \circ y &= (-1)^x y \mod 2n \\ x \bullet y &= (-1)^{y+1} x + x + y \mod 2n \end{aligned}$$

is left distributive but not right distributive.

2-reductivity $\Longrightarrow$ distributivity

Definition

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-reductive if it satisfies for $x, y, z \in X$:

$$(x \circ y) \circ z = y \circ z$$
$$z \bullet (y \bullet x) = z \bullet y$$
$$(y \bullet x) \circ z = y \circ z$$
$$z \bullet (x \circ y) = z \bullet y$$

Corollary

Each 2-reductive birack is distributive.

Retraction relations of a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio)

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - *a distributive birack*

- $(X/\sim, \circ, \backslash_\circ, \bullet, /_\bullet) = \mathbf{B}_L(X/\sim, \circ, \backslash_\circ)$
- $(X/\backsim, \circ, \backslash_\circ, \bullet, /_\bullet) = \mathbf{B}_R(X/\backsim, \bullet, /_\bullet)$
- $(X/\approx, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is idempotent*

$$x \sim y \quad \Leftrightarrow \quad L_x = L_y \quad \Leftrightarrow \quad \forall a \in X \quad x \circ a = y \circ a$$

$$x \backsim y \quad \Leftrightarrow \quad \mathbf{R}_x = \mathbf{R}_y \quad \Leftrightarrow \quad \forall a \in X \quad a \bullet x = b \bullet y$$

$$x \approx y \quad \Leftrightarrow \quad L_x = L_y \quad \text{and} \quad \mathbf{R}_x = \mathbf{R}_y$$

Distributive biracks

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio)

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - *a distributive birack*

- $[\mathrm{LMlt}(X), \mathrm{RMlt}(X)] = \{\mathrm{id}\}$
- *The group* $\mathrm{Mlt}(X)$ *is abelian* $\Leftrightarrow$ $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is 2-reductive.*

k-reductive biracks, $k \geq 2$

Definition

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is k-**reductive** if it is

left k-reductive

$$(\ldots((x_0 \circ x_1) \circ x_2) \ldots) \circ x_k = (\ldots((x_1 \circ x_2) \circ x_3) \ldots) \circ x_k,$$

and **right k-reductive**

$$x_0 \bullet (\ldots(x_{k-2} \bullet (x_{k-1} \bullet x_k)) \ldots) = x_0 \bullet (\ldots(x_{k-2} \bullet x_{k-1})) \ldots)$$

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio)

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is k-reductive* $\Leftrightarrow$
the group $\mathrm{Mlt}(X)$ *is nilpotent of class at most* $k-1$.

k-reductive biracks

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio)

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *- a distributive birack and* $k \geq 2$

The following conditions are equivalent:

- $|\mathrm{Ret}^k(X)| = 1$,
- $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is k-reductive,*
- $\mathrm{Mlt}(X)$ *is nilpotent of class at most* $k-1$.

Remark

The Theorem cannot be generalized for non-distributive biracks.

- There exist biracks that are reductive but their multiplication groups are not nilpotent (A. Smoktunowicz)
- There exist biracks that are not reductive but their multipication groups are nilpotent (A. Smoktunowicz)

Distributive biracks $\neq$ 2-reductive

Example

$(\{1,2,3,4,5,6\}, \circ, \backslash_\circ, \bullet, /_\bullet)$ - a distributive birack such that:

$$L_1 = (3546) \qquad L_2 = (6453)$$
$$L_3 = L_4 = (12)(56) \qquad L_5 = L_6 = (12)(34)$$
$$\mathbf{R}_1 = \mathbf{R}_2 = \mathrm{id} \qquad \mathbf{R}_3 = \mathbf{R}_4 = \mathbf{R}_5 = \mathbf{R}_6 = (34)(56)$$

$$5 = 5 \circ 5 = (1 \circ 3) \circ 5 \neq 3 \circ 5 = 6$$

Distributive involutive biracks

Theorem

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - *involutive birack*

The following conditions are equivalent:

1. $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is distributive*
2. $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is 2-reductive*

A birack is *involutive* if for every $x, y \in X$:

$$(x \circ y) \circ (x \bullet y) = x,$$
$$(x \circ y) \bullet (x \bullet y) = y$$

equivalently: $x \bullet y = L^{-1}_{x\circ y}(x) = (x \circ y)\backslash_\circ x$ and $x \circ y = \boldsymbol{R}^{-1}_{x\bullet y}(y) = y/_\bullet(x \bullet y)$

Distributivity:

$$(x \circ y) \circ z = y \circ z$$
$$z \bullet (y \bullet x) = z \bullet y$$
$$(y \bullet x) \circ z = y \circ z$$
$$z \bullet (x \circ y) = z \bullet y$$

The Structure Theorem

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio, 2020)

Each involutive distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is a disjoint union of abelian groups* $((A_i)_{i\in I};\ (c_{i,j})_{i,j\in I})$, *with operations for* $x \in A_i$ *and* $y \in A_j$*:*

$$x \circ y = y + c_{i,j} \quad \text{and} \quad x \bullet y = x - c_{j,i},$$

where $A_j = \langle \{c_{i,j} \mid i \in I\} \rangle$, *for every* $j \in I$.

W. Rump, 2022: The *Transvection torsor*

Involutive 2-reductive biracks of size 3

Example

- One orbit: $((\mathbb{Z}_3), (1))$.
- Two orbits: $((\mathbb{Z}_2, \mathbb{Z}_1), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_1), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $((\mathbb{Z}_2, \mathbb{Z}_1), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$.
- Three orbits: $((\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$.

(Non-involutive) 2-reductive biracks of size 3

Example

- One orbit: $(\mathbb{Z}_3, (1), (1))$, $(\mathbb{Z}_3, (0), (1))$, $(\mathbb{Z}_3, (1), (0))$.
- Two orbits: $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$ $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$, $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))$, $(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$,
$(\mathbb{Z}_2 \cup \mathbb{Z}_1, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right))$.
- Three orbits: $(\mathbb{Z}_1 \cup \mathbb{Z}_1 \cup \mathbb{Z}_1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$.

2-permutational biracks

Definition

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is 2-permutational if for every $x, y, z, t \in X$:

$$(z \circ x) \circ y = (t \circ x) \circ y \quad \Leftrightarrow \quad L_{L_z(x)} = L_{L_t(x)}$$
$$(x \bullet z) \circ y = (x \bullet t) \circ y \quad \Leftrightarrow \quad L_{\boldsymbol{R}_z(x)} = L_{\boldsymbol{R}_t(x)}$$
$$y \bullet (z \circ x) = y \bullet (t \circ x) \quad \Leftrightarrow \quad \boldsymbol{R}_{L_z(x)} = \boldsymbol{R}_{L_t(x)}$$
$$y \bullet (x \bullet z) = y \bullet (x \bullet t) \quad \Leftrightarrow \quad \boldsymbol{R}_{\boldsymbol{R}_z(x)} = \boldsymbol{R}_{\boldsymbol{R}_t(x)}$$

2-reductive $\neq$ 2-permutational

2-reductive $\neq$ 2-permutational

Example

$(X = \{0, 1, 2, 3\}, \circ, \backslash_\circ, \bullet, /_\bullet)$ - 2-permutational involutive birack:

$\circ$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

$\bullet$	0	1	2	3
0	3	1	3	1
1	2	0	2	0
2	1	3	1	3
3	0	2	0	2

$$2 = 2 \circ 3 = (0 \circ 3) \circ 3 \neq 3 \circ 3 = 0$$

$(X,*,\backslash_*,\diamond,/_\diamond)$: σ-isotope of a birack $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$

$(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ - a 2-reductive involutive birack

σ - a bijection on the set X such that

$$\forall x,y\in X \quad L_{\sigma(y)}\sigma L_x = L_{\sigma(x)}\sigma L_y$$

$$x*y = L_x\sigma(y) \quad \text{and} \quad x\diamond y = (x*y)\backslash_* x$$

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio)

The σ-isotope $(X,*,\backslash_*)$ of $(X,\circ,\backslash_\circ)$ is such that:

- *right cyclic left quasigroup*
- *non-degenerate*
- $(z*x)*y = (t*x)*y$

Then $(X,*,\backslash_*,\diamond,/_\diamond)$ is a 2-permutational involutive birack.

[W.Rump] An algebra $(X,\circ,\backslash_\circ,\bullet,/_\bullet)$ is an involutive birack if and only if $(X,\circ,\backslash_\circ)$ is a non-degenerate right cyclic left quasigroup.

Example

Involutive 2-reductive

$\circ$	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	0	1	2	3
3	2	3	0	1

$\sigma = (01)(23)$ satisfies the condition: $L_{\sigma(y)}\sigma L_x = L_{\sigma(x)}\sigma L_y$

σ-isotope = 2-permutational involutive

$\circ$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

2-permutational involutive biracks

Lemma

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - *a* 2*-permutational involutive birack*

$e \in X$, $L_e^{-1} \neq \mathrm{id}$

The L_e^{-1}*-isotope of* $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is an involutive* 2*-reductive birack.*

Example

Involutive 2-permutational non-distributive

$\circ$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

L_0^{-1}-isotope = involutive 2-reductive

$\circ$	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	0	1	2	3
3	2	3	0	1

Example

2-permutational involutive

$\circ$	0	1	2	3	4
0	0	2	1	4	3
1	3	2	1	0	4
2	4	2	1	3	0
3	0	2	1	4	3
4	0	2	1	4	3

L_0^{-1} and L_1^{-1}- isotopes = 2-reductive, involutive

$*_0$	0	1	2	3	4
0	0	1	2	3	4
1	3	1	2	4	0
2	4	1	2	0	3
3	0	1	2	3	4
4	0	1	2	3	4

$\ncong$

$*_1$	0	1	2	3	4
0	4	1	2	0	3
1	0	1	2	3	4
2	3	1	2	4	0
3	4	1	2	0	3
4	4	1	2	0	3

2-permutational involutive biracks

Theorem (P. Jedlička, A.P., A. Zamojska-Dzienio)

Each 2-permutational involutive birack is a π-isotope of a 2-reductive one, for some bijection π.

Proof - Main idea

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ - a 2-permutational involutive birack, $e \in X$

L_e^{-1}-isotope = $(X, *, \backslash_*, \diamond, /_\diamond)$ = 2-reductive involutive birack

$$x * y = L_x L_e^{-1}$$

L_e satisfies the condition: $L^*_{\sigma(y)} \sigma L^*_x = L^*_{\sigma(x)} \sigma L^*_y$ for $(X, *, \backslash_*, \diamond, /_\diamond)$

Construct L_e-isotope of $(X, *, \backslash_*, \diamond, /_\diamond)$ = $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$

Enumeration

n	1	2	3	4	5	6	7	8
involutive biracks	1	2	5	23	88	595	3456	34528
2-permutational inv. biracks	1	2	5	19	70	359	2095	16332
2-reductive inv. biracks	1	2	5	17	65	323	1960	15421
2-permutational, not 2-reductive	0	0	0	2	5	36	135	911

Table: The number of involutive biracks of size n, up to isomorphism.

THANK YOU VERY MUCH FOR YOUR ATTENTION!