Biracks and their applications I

Přemysl Jedlička[1], Agata Pilitowska[2], Anna Zamojska-Dzienio[2]

[1] Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences,
[2] Faculty of Mathematics and Information Science, Warsaw University of Technology

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Będlewo

Introduction

- Quandles and racks
- Biracks and biquandles
- Solutions of the Yang–Baxter equation

(Mathematical) knots

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- Various **knot invariants** $=$ functions that have the same outcome for two equivalent knots.

Quandles

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- associates a quandle with every knot diagram - the **knot quandle**. This is a complete knot invariant: when two knots have isomorphic knot quandles, they can differ only in orientation.

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J.S. Carter et al. 2004: quandle cohomology.

J. S. Carter, M. Elhamdadi, M. Saito, *Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles*, Fund. Math. 184 (2004), 31–54.

Quandles

An algebra $(Q, *, /)$, with two binary operations $*$ and $/$, is a **quandle** if for every $x, y, z \in Q$:

1. $x * x = x$
2. $(x * y)/y = x = (x/y) * y$
3. $(y * z) * x = (y * x) * (z * x)$

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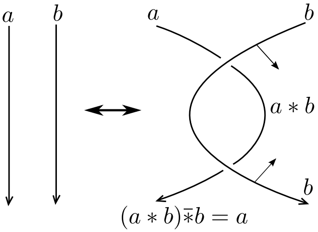
Right quasigroup: the equation $u * x = y$ has a unique solution $u \in Q$ for every $x, y \in Q$.

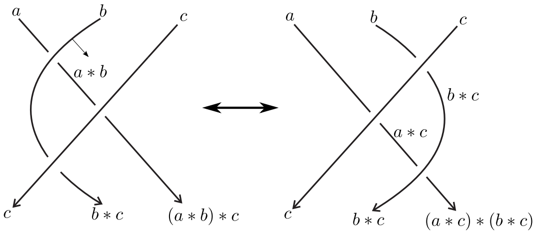
All **right translations** $R_a \colon Q \to Q; \;\; R_a(x) = x * a$ are automorphisms of Q.

Racks

2nd Reidemeister move



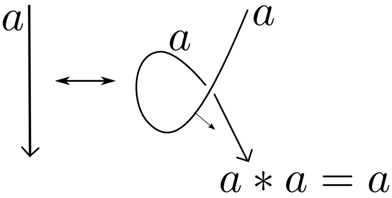
3rd Reidemeister move



Right distributive right quasigroup = racks

Quandles

1st Reidemeister move



Idempotent racks = quandles

Biracks

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- R. Fenn, M. Jordan-Santana, L. Kauffman, *Biquandles and virtual links*, Topology and its Appl. **145** (2004), 157–175.
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Biracks

Quandles and racks (Joyce 1982, Matveev 1984) $\leftrightarrow$ (classical) knot theory.

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Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a birack, where for every $x, y, \in X$:

$$x \circ y = y.$$

Then $(X, \bullet, /_\bullet)$ is a (right) rack.

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Equational characterization

Definition (Stanovský 2006)

A structure $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ with four binary operations is called a **birack**, if the following holds for any $x, y, z \in X$:

$$\begin{aligned} x \circ (x \backslash_\circ y) = y &= x \backslash_\circ (x \circ y), \\ (y /_\bullet x) \bullet x = y &= (y \bullet x) /_\bullet x, \\ x \circ (y \circ z) &= (x \circ y) \circ ((x \bullet y) \circ z), \\ (x \circ y) \bullet ((x \bullet y) \circ z) &= (x \bullet (y \circ z)) \circ (y \bullet z), \\ (x \bullet y) \bullet z &= (x \bullet (y \circ z)) \bullet (y \bullet z). \end{aligned}$$

D. Stanovský, *On axioms of biquandles*, J. Knot Theory Ramifications **15** (2006), 931–933.

Involutive biracks

A birack is **involutive** if it satisfies for every $x, y \in X$:

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Then one has

$$x \bullet y = (x \circ y) \backslash_{\circ} x,$$
$$x \circ y = y /_{\bullet} (x \bullet y).$$

Multiplication groups

Multiplication groups

In a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$:

left translations: $L_x\colon X \to X$ by x; $L_x(a) = x \circ a$ and

right translations: $\mathbf{R}_x\colon X \to X$ by x; $\mathbf{R}_x(a) = a \bullet x$

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Three types of multiplication groups:

$\mathrm{LMlt}(X) = \langle L_x : \ x \in X \rangle$

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For involutive biracks these groups are equal.

Distributive biracks

P. Jedlička, A. Pilitowska, A. Zamojska-Dzienio, *Distributive biracks and solutions of the Yang-Baxter equation*, Internat.J.Algebra Comput. **30** (2020), 667–683.

Distributive biracks

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is **left distributive**, if for every $x, y, z \in X$:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z),$$

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$$(y \bullet z) \bullet x = (y \bullet x) \bullet (z \bullet x).$$

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The birack is **distributive** if it is left and right distributive.

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An involutive birack is left distributive iff it is right distributive.

P. Jedlička, A. Pilitowska, A. Zamojska-Dzienio, *Distributive biracks and solutions of the Yang-Baxter equation*, Internat.J.Algebra Comput. **30** (2020), 667–683.

Examples

Permutational birack (Lyubashenko (Drinfeld 1992)):

$X \neq \varnothing$, $f, g \colon X \to X$ bijections such that $fg = gf$.

Define operations:

$$x \circ y = f(y),\ x \backslash_\circ y = f^{-1}(y),$$
$$x \bullet y = g(x),\ x /_\bullet y = g^{-1}(x).$$

Then $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a **permutational birack**.

It is involutive iff $g = f^{-1}$ ($\leftrightarrow$ *permutation solution*).

If $f = g = \mathrm{id}$, the birack is a **projection** one ($\leftrightarrow$ *trivial solution*).

Examples

Derived biracks

$(X, \circ, \backslash_\circ)$ left rack.

Define operations $\bullet, /_\bullet \colon X \times X \to X$ as $x \bullet y = x = x /_\bullet y$. Then the structure $\mathbf{B}_L(X, \circ, \backslash_\circ) = (X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a **left derived** **birack**.

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For a left rack $(X, *, \backslash_*)$ and a right rack $(Y, \triangle, /_\triangle)$, the product

$$\mathbf{B}_L(X, *, \backslash_*) \times \mathbf{B}_R(Y, \triangle, /_\triangle)$$

is a distributive birack with $\mathrm{Mlt}(X \times Y) \cong \mathrm{LMlt}(X) \times \mathrm{RMlt}(Y)$.

Biquandles

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Each involutive birack is a biquandle.

Direct calculations: P. Jedlička, A. Pilitowska, A. Zamojska-Dzienio, *The retraction relation for biracks*, J. Pure Appl. Algebra **223** (2019), 3594–3610.

Examples

Wada switch

$(G, \cdot, e)$ a group. Define operations:

$$x \circ y = xy^{-1}x^{-1},\ x \backslash_\circ y = x^{-1}y^{-1}x,$$
$$x \bullet y = xy^2,\ x /_\bullet y = xy^{-2}.$$

The birack $(G, \circ, \backslash_\circ, \bullet, /_\bullet)$ known as the **Wada switch** or **Wada biquandle** (Fenn et al.).

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$(G, \circ, \backslash_\circ, \bullet, /_\bullet)$ is left distributive iff $y^2 \in Z(G)$, for all $y \in G$, and is right distributive iff $x^4 = e$ and $x^2 \in Z(G)$, for all $x \in G$.

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It is involutive iff $(G,\cdot,e)$ is an elementary abelian 2-group.

Distributivity - alternative characterization

Proposition (P. Jedlička et al. 2020)

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *be a structure with four binary operations. Then* $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is a distributive birack if and only if the following conditions are satisfied*

(i) $(X, \circ, \backslash_\circ)$ *is a left rack and* $(X, \bullet, /_\bullet)$ *is a right rack,*

(ii) *for all* $x, y, z \in X$

$$(x \bullet y) \circ z = x \circ z,$$
$$x \bullet (y \circ z) = x \bullet z,$$
$$x \circ (y \bullet z) = (x \circ y) \bullet z.$$

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Multiplication groups for distributivity

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Proposition (P. Jedlička et al. 2020)

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a distributive birack. Then, for each $x \in X$, the bijections L_x and $\mathbf{R}_x$ are automorphisms of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

Groups $\mathrm{LMlt}(X)$, $\mathrm{RMlt}(X)$, $\mathrm{Mlt}(X)$ are normal subgroups of the automorphism group of a distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

Yang-Baxter equation

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Let V be a vector space. A **solution of the Yang–Baxter equation** is a linear mapping $r : V \otimes V \to V \otimes V$ such that

$$(id \otimes r)(r \otimes id)(id \otimes r) = (r \otimes id)(id \otimes r)(r \otimes id).$$

First introduced in the field of statistical mechanics: **C. N. Yang 1968**, and **R. J. Baxter 1971**.

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Profound implications for many areas of mathematics and physics:

1. how waves behave in shallow water,
2. the interaction of subatomic particles,
3. the mathematical theory of knots,
4. string theory, ...

Simplifications

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Drinfeld 1992: **set-theoretical solution of the Yang–Baxter equation** (X, σ, τ).

V.G. Drinfeld, *On some unsolved problems in quantum group theory*, In: P.P. Kulish (ed.) Quantum groups, in: Lecture Notes in Math., vol. 1510, Springer-Verlag, Berlin, 1992, pp. 1–8.

Simplifications

Drinfeld 1992: **set-theoretical solution of the Yang–Baxter equation** (X, σ, τ).

- Let X be a basis of the space V and let $\sigma : X^2 \to X$ and $\tau : X^2 \to X$ be two mappings such that the mapping $x \otimes y \mapsto \sigma(x, y) \otimes \tau(x, y)$ extends to a solution of the Yang–Baxter equation.

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(Set-theoretical) solutions of YBE

P. Etingof, T. Schedler, A. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. **100** (1999), 169–209.

(Set-theoretical) solutions of YBE

It means that $r\colon X^2 \to X^2$, where $r = (\sigma, \tau)$ satisfies the **braid relation**:

$$(id \times r)(r \times id)(id \times r) = (r \times id)(id \times r)(r \times id).$$

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$(id \times r)(x, y, z) = (x, r(y, z)) = (x, \sigma(y, z), \tau(y, z))$

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A solution is:

- **non-degenerate** if the mappings $\sigma(x, \_) = \sigma_x\colon X \to X$ and $\tau(\_, y) = \tau_y\colon X \to X$ are bijections, for all $x, y \in X$;
- **involutive** if $r^2 = \mathrm{id}_{X^2}$;
- **square-free** if $r(x, x) = (x, x)$, for every $x \in X$.

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Solutions vs. biracks

Fenn et al. 2004: There is a one-to-one correspondence between non-degenerate solutions of the Yang-Baxter equation (X, σ, τ) and **biracks** $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$:

$$r(x, y) = (\sigma(x, y), \tau(x, y)) = (x \circ y, x \bullet y) = (L_x(y), \mathbf{R}_y(x)).$$

- R. Fenn, M. Jordan-Santana, L. Kauffman, *Biquandles and virtual links*, Topology and its Appl. **145** (2004), 157–175.

Solutions vs. biracks

Fenn et al. 2004: There is a one-to-one correspondence between non-degenerate solutions of the Yang-Baxter equation (X, σ, τ) and **biracks** $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$:

$$r(x, y) = (\sigma(x, y), \tau(x, y)) = (x \circ y, x \bullet y) = (L_x(y), \mathbf{R}_y(x)).$$

- R. Fenn, M. Jordan-Santana, L. Kauffman, *Biquandles and virtual links*, Topology and its Appl. **145** (2004), 157–175.
- P. Dehornoy, *Set-theoretic solutions of the Yang-Baxter equation, RC-calculus, and Garside germs*, Adv. Math. **282** (2015), 93–127.

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In an involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$:

$$x \bullet y = (x \circ y)\backslash_\circ x,$$
$$x \circ y = y /_\bullet (x \bullet y).$$

Cycle-sets

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Biquandles...

Non-deg. cycle-sets vs. inv. biracks

If a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is involutive, then $(X, \backslash_\circ, \circ)$ is a non-degenerate cycle-set.

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If a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is involutive, then $(X, \backslash_\circ, \circ)$ is a non-degenerate cycle-set.

Theorem (Rump 2005, Dehornoy 2015)

Let $(X, \backslash, *)$ be a non-degenerate cycle-set. Then defining $x \circ y = x * y$, $x \backslash_\circ y = x \backslash y$, $x \bullet y = (x * y) \backslash x$, and $x /_\bullet y = z$, where z is the unique one such that $z \backslash z = y * (x \backslash x)$, the algebra $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is an involutive birack.

Congruence on a birack

Definition

An equivalence relation θ on the set X of elements of a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a **congruence on** $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ if it is compatible with all four operations of the birack X, i.e. if $x \,\theta\, y$ and $z \,\theta\, t$ then also

$$(x \circ z) \,\theta\, (y \circ t)$$
$$(x \backslash_\circ z) \,\theta\, (y \backslash_\circ t)$$
$$(x \bullet z) \,\theta\, (y \bullet t)$$
$$(x /_\bullet z) \,\theta\, (y /_\bullet t).$$

Quotient birack

If θ is a congruence on a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$, then the quotient set $X^\theta = \{x^\theta : x \in X\}$ of the equivalence classes under θ, is again a birack, called the **quotient birack**, under operations defined by

$$
\begin{aligned}
x^\theta \circ z^\theta &= (x \circ z)^\theta \\
x^\theta \backslash_\circ z^\theta &= (x \backslash_\circ z)^\theta \\
x^\theta \bullet z^\theta &= (x \bullet z)^\theta \\
x^\theta /_\bullet z^\theta &= (x /_\bullet z)^\theta .
\end{aligned}
$$

Generalized retraction congruence

Definition

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a birack. The equivalence relation $\approx$ defined on X in the following way

$$x \approx y \Leftrightarrow L_x = L_y \text{ and } \mathbf{R}_x = \mathbf{R}_y$$

is called the **generalized retraction**.

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Theorem (P. Jedlička et al. 2019)

The generalized retraction is a congruence of a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

P. Jedlička, A. Pilitowska, A. Zamojska-Dzienio, *The retraction relation for biracks*, J. Pure Appl. Algebra **223** (2019), 3594–3610.

Retraction relations

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ a birack.

Three retraction relations:

$$
\begin{aligned}
a \sim b \quad &\Leftrightarrow \quad L_a = L_b \quad \Leftrightarrow \quad \forall x \in X \quad a \circ x = b \circ x, \\
a \backsim b \quad &\Leftrightarrow \quad \mathbf{R}_a = \mathbf{R}_b \quad \Leftrightarrow \quad \forall x \in X \quad x \bullet a = x \bullet b, \\
a \approx b \quad &\Leftrightarrow \quad a \sim b \wedge a \backsim b \quad \Leftrightarrow \quad L_a = L_b \wedge \mathbf{R}_a = \mathbf{R}_b.
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They are congruences (and they are equal) for an **involutive** birack.

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They are congruences (and they are equal) for an **involutive** birack.

They are congruences for a **distributive** birack.

Retracts

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ a birack.

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iterated retraction: $\mathrm{Ret}^0(X) = (X, \circ, \backslash_\circ, \bullet, /_\bullet)$ and $\mathrm{Ret}^k(X) = \mathrm{Ret}(\mathrm{Ret}^{k-1}(X))$, for any natural number $k > 1$.

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A birack is of **multipermutation level** k, if $|\mathrm{Ret}^k(X)| = 1$ and $|\mathrm{Ret}^{k-1}(X)| > 1$.

Retraction and distributivity

Theorem (P. Jedlička et al. 2020)

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *be a distributive birack and let* $k \geq 2$. *Then the following conditions are equivalent:*

(i) $|\mathrm{Ret}^k(X)| = 1$,

(ii) $\mathrm{Mlt}(X)$ *is nilpotent of class at most* $k-1$.

Retraction relation

Etingof et al. 1999:

P. Etingof, T. Schedler, A. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. **100** (1999), 169–209.

Retraction relation

Etingof et al. 1999:

- for non-degenerate **involutive** (X, σ, τ) the equivalence relation $\sim$ on the set X: for each $x, y \in X$

$$x \sim y \quad \Leftrightarrow \quad \tau(\_, x) = \tau(\_, y).$$

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- a **multipermutation solution of level k**, if k is the smallest integer such that $|\mathrm{Ret}^k(X)| = 1$.

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THANK YOU!