Minimal associativity in quasigroups

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Outline

Maximally nonassociative quasigroups

Associativity index

For a quasigroup Q denote by $a(Q)$ the number of *associative triples*, that is the number of $(x, y, z) \in Q^3$ such that $x \cdot yz = xy \cdot z$. Call $a(Q)$ the *associativity index* of Q.

Some associative triples

Denote by e_x and f_x the *local units* of Q. Thus $e_x x = x$ and $xf_x = x$. Then $e_x x \cdot f_x = xf_x = x = e_x x = e_x \cdot xf_x$. This means that (e_x, x, f_x) is always an associative triple. Hence $a(Q) \geq |Q|$.

Problem: Kepka, 1981

Does there exist a finite nontrivial quasigroup Q such that $a(Q) = |Q|$?

Conjecture: Grošek and Horák, 2012

There is no finite nontrivial quasigroup with $a(Q) = |Q|$.

Maximally nonassociative quasigroups (mnqs)

When Q is a mnq—a definition and an easy fact

Def. of a mnq: $(x, y, z) \in Q^3$ is associative $\iff$ $x = y = z$, $|Q| \geq 2$.
Lemma: $a(Q) = |Q| < \infty \Rightarrow Q$ idempotent (proof later). Thus a mnq.

Constructions of maximally nonassociative quasigroups

2017 Valent: Computer finds a mnq of order 9;
2018 Lisoněk: Many mnqs follow from nearfields;
2019 Wanless: To get primes use quadratic orthomorphisms;
2019 Drápal: Combine mnqs by a product construction.

Existence and nonexistence of a mnq of order n

- no mnq exists if $2 \leq n \leq 8$ or $n = 10$;
- no mnq known if $n = 2p$ or $n = 2p_1p_2$, $p_1 \leq p_2 < 2p_1$;
- no mnq known if $n \in \{11, 12, 15, 40, 42, 44, 56, 66, 77, 88, 90, 110\}$;
- for all other n there exists a mnq.

Road to the nearfield construction

What was found by computer

	1	2	3	4	5	6	7	8	9
1	1	4	7	3	8	5	2	9	6
2	8	2	5	6	1	9	4	3	7
3	6	9	3	7	4	2	8	5	1
4	5	3	9	4	7	1	6	2	8
5	7	6	1	2	5	8	9	4	3
6	2	8	4	9	3	6	1	7	5
7	9	5	2	8	6	3	7	1	4
8	3	7	6	1	9	4	5	8	2
9	4	1	8	5	2	7	3	6	9

Up to isomorphism the only quasigroup of order 9 that is maximally nonassociative.
It yields a Sudoku square.

Intepretative efforts

Drápal gave an interpretation via an affine plane. This resulted in computer experiments (Kozlik, Lisoněk) that led to the discovery of intepretation in the form $x * y = x + (y - x) \circ c$, where $(N, +, \circ, 0, 1)$ is a nearfield and $c \in N$, $c \notin \{0, 1\}$.

About nearfields

Definition of a (left) nearfield $(N, +, \circ, 0, 1)$

1. $(N, +, 0)$ is an Abelian group;
2. $(N^*, \circ, 1)$ is a group, where $N^* = N \setminus \{0\}$;
3. $x \circ 0 = 0 = 0 \circ x$ for all $x \in N$; and
4. $x \circ (y + z) = x \circ y + x \circ z$ for all $x, y, z \in N$.

Classification and Dickson's nearfields

Finite nearfields are completely classified (Zassenhaus). *Dickson's nearfields* are defined on $\mathbb{F}_{q^2}$, q power of an odd prime so that
$x \circ y = xy$ **if** x a **square**, $x \circ y = xy^q$ **if** x a **nonsquare**.

Quasigroups derived from a nearfield (Stein)

If $c \neq 0, 1$, then $x *_c y = x + (y - x) \circ c$ is a quasigroup. The mappings $x \mapsto \lambda \circ x$ and $x \mapsto x + u$ are automorphisms of $(N, *_c)$ for all $\lambda \in N^*$ and $u \in N$. All quasigrps Q with $\mathrm{Aut}(Q)$ sharply 2-transitive are of this form.

Under which conditions gives a nearfield a mnq?

A useful lemma

Q idempotent quasigroup. If (x, x, y) or (y, x, x) ass., then $x = y$.

A consequence for quasigroups with $\mathrm{Aut}(Q)$ 2-transitive

Let $0, 1 \in Q$. Then Q is a mnq $\Leftrightarrow$ $(0, 1, z)$ associative for no $z \in Q$.

What does this mean for quasigroups over nearfields?

$0 *_c z = 0 + (z - 0) \circ c = z \circ c$. Thus $0 * (1 * z) = (1 * z) \circ c$ gives $(1 + (z - 1) \circ c) \circ c$, while $(0 * 1) * z = c * z = c + (z - c) \circ c$.

Multiply to simplify: $x \circ (z - 1) = 1$

$z - c = 1 + (z - 1) - c$, $x \circ (c + (z - c) \circ c) = \underline{x \circ c + (x + 1 - x \circ c) \circ c}$ and $x \circ (1 + (z - 1) \circ c) \circ c = \underline{(x + c) \circ c}$.

When Dickson's nearfield gives a mnq?

$x \circ c + (x + 1 - x \circ c) \circ c = (x + c) \circ c$

should never hold. Set ε_0, ε_1, ε_2 to zero if x, $x + 1 - x \circ c$, $x + c$ a square. Otherwise $\varepsilon_i = 1$. Interpret the equation for each choice $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$.

$\varepsilon = (0, 0, 0)$ and $\varepsilon = (0, 0, 1)$

$xc + (x + 1 - xc)c = (x + c)c \Leftrightarrow c(x + 1)(c - 1) = 0$
$xc + (x + 1 - xc)c = (x + c)c^q \Leftrightarrow c(x(c^{q-1} + c - 2) + c^q - 1) = 0$

What does it say?

$\varepsilon = (0, 0, 0)$: one of -1, c and $-1 + c$ has to be a nonsquare.
$\varepsilon = (0, 0, 1)$: not mnq if $x(c^{q-1} + c - 2) = (1 - c)^q$, x a square, and ... If so, $x^{-1} = ((c^{q-1} + c - 2)/(1 - c)^q = ((1 - c)^{2-q} - 1)c^{-1}$ is a square, and thus $((1 - c)^{2-q} - 1)c$ is a square. Employing values ε_1 and ε_2 shows that $(\mathbb{F}_{q^2}, *_c)$ not mnq if $c^{q-1}(c^2 - c + 1) - 1$ a nonsquare.

Squares, nonsquares and Weil's bound

Six other terms that should avoid zero

$c^2(xc^{q-2}(c-1)-c^{q-2}-1)$, $c(c-1)(x(c-1)^{q-1}+c^{q-1})$, $c(c-1)(xc^{q-1}+1)$, $c(c-1)^q(x+1)$, $c(x(2c^{q-1}-c^{2q-1}-1)+c^q-1-c)$, $c^q(c-1)(x(c-1)^{q-1}+1)$.

The result of processing the conditions

A mnq $(\mathbb{F}_{q^2}, *_c)$ exists iff $\cup K_i$ a <u>proper</u> subset of $\mathbb{F}_{q^2}$. K_i are sets of $c \in \mathbb{F}_{q^2}$, $c \notin \{0,1\}$, where

K_0: c and $c-1$ are squares,

K_1: $c-1$ and $c^{q-1}(c^2-c+1)-1$ are nonsquares, $c((c-1)^{2-q}+1)$ is a square,

K_2: c, $(c+1)-(c-1)^{q-1}$ and $c^{-1}-(c^{-1}-1)^{q-1}$ are nonsquares,

K_3: $(c^{-1}-1)^{q-1}+(c-1)$ and $c(c^{-1}-1)^{q-1}-1$ nonsquares,

K_4: $(c^{2q}-2c^q+c)(c^q-c^2)$ and $c^q(c^{q+1}-2c+1)(c^q-c^2)$ are nonsquares, while $(c-1)(c^{q+1}+c^q-c)(c^q-c^2)$ is a square.

Weil's bound: r polynomial sq/nsq conditions $\approx |\mathbb{F}|/2^r$ solutions

This is approximative, size of error depends on polynomial degrees. To get upper estimates of $|K_i|$ the polynomials are thus turned into polynomials of small degree in two variables over $\mathbb{F}_q$.

Final slide on quasigroups from nearfields

Results

Applying Weil's bound to the list of polynomials: For $q > 14400$ there are not enough c that fulfil at least one ε condition. Hence for $q > 14400$ there exists c for which $(\mathbb{F}_{q^2}, *_c)$ is maximally nonassociative.
Computer: Such a c exists for each $q < 14400$ too.
Computations suggest that $(\mathbb{F}_{q^2}, *_c)$ is a mnq with limit probability 0.289.
In every proper nearfield N, $|N| < 10000$, there $\exists\ c \in N$ such that $(N, *_c)$ is a mnq.

Source

Drápal & Lisoněk, *Maximal nonassociativity via nearfields*, Finite Fields and Their Applications **62** (2020).

Product constructions

Direct product

Q_1 and Q_2 quasigroups: $a(Q_1 \times Q_2) = a(Q_1)a(Q_2)$.
Hence Q_1 and Q_2 mnqs $\Rightarrow$ $Q_1 \times Q_2$ is a mnq as well.

Product construction using idempotent quasigroups

Result: *Let $(Q, \cdot)$ and $(U, *)$ be finite quasigroups, $|Q| \geq |U|$ and U idempotent. Then there exists a quasigroup on $Q \times U$, the associativity index of which is equal to $a(Q) \cdot |U|$.*

We need $j\colon U \to Q$ injective mapping and $(Q, +)$ abelian group. Let

$$(x,\, u)(y,\, v) = \begin{cases} (x \cdot y,\, u) & \text{if } u = v \text{, and} \\ (x + y + j(u),\, u * v) & \text{if } u \neq v. \end{cases}$$

A consequence for maximally nonassociative quasigroups

If $n \geq m > 2$ and $\exists$ a mnq of order n, then $\exists$ a mnq of order mn.
This is because an idempotent quasigroup $\exists$ for each $m \geq 3$.

Elementary associative triples

Triples (e_y, y, z), (x, y, f_y) and $(x, f_x = e_z, z)$.

Q a quasigroup with local units e_x and f_x.

$e_y = e_{yz} \;\Rightarrow\; e_y \cdot yz = yz = e_y y \cdot z$;

$f_y = f_{xy} \;\Rightarrow\; xy \cdot f_y = xy = x \cdot yf_y$;

$y = f_x = e_z \;\Rightarrow\; xy \cdot z = xz = x \cdot yz$.

These associative triples are called *elementary*. Criterion:

An associative triple (x, y, z) is elementary $\iff xyz \in \{xy, xz, yz\}$.

Grošek—Horák inequality

$a(Q) \geq 2|Q| - |\mathrm{I}(Q)|$, where $\mathrm{I}(Q)$ is the set of idempotents of Q.

A consequence of the inequality

$a(Q) = |Q| \;\Rightarrow\; \mathrm{I}(Q) = Q$.

Finite maximally nonassociative quasigroups are idempotent.

Extremely nonassociative quasigroups (enqs)

Definition of a finite exnq

Q is extremely nonassociative $\iff$ $a(Q) = 2|Q| - |\mathrm{I}(Q)|$, $|Q| \geq 2$.

Improved Grošek—Horák inequality

$a(Q) \geq 2|Q| - |\mathrm{I}(Q)| + \delta(Q)$, where $\delta(Q)$ is the number of fixed point free left translations L_x plus the number of fixed point free right translations R_x. (The proof is quite long, cf. D & Valent, JCD 2018.)

Consequences for a finite exnq Q

- Mappings $e\colon x \mapsto e_x$ and $f : x \mapsto f_x$ permute Q; and
- The only associative triples of Q are $(e(x), x, f(x))$ and $(e^{-1}(x), x, f^{-1}(x))$, $x \in Q$.

This can be used as a definition of an exnq that covers infinite Q too.

Existence of extremely nonassociative quasigroups

Orders 8 and 9

Up to $\cong$ 6 exnqs of order 8, forming 3 pairs of opposite quasigroups and belong to two main classes. Associativity index $= 16$ (no idempotents).
Up to $\cong$ 3 exnqs of order 9. One is the mnq. The other two mirror each other and have 17 associative triples (one idempotent).

Extremely nonassociative quasigroup of order eight

	1	2	3	4	5	6	7	8
1	3	8	2	4	6	1	5	7
2	1	4	5	3	8	7	2	6
3	4	2	1	6	7	5	8	3
4	7	1	3	2	4	8	6	5
5	5	7	6	1	3	2	4	8
6	2	6	8	7	5	4	3	1
7	8	3	7	5	2	6	1	4
8	6	5	4	8	1	3	7	2

$e = (1\,2\,3\,4)(5\,6\,7\,8)$,
$f = (1\,6\,2\,7\,3\,8\,4\,5)$.

To get orders > 8 direct product not applicable. The other product construction yields exnqs of orders $2^k m$, where $m < 2^k$ is odd and $k \geq 3$, $k \neq 4$

For which orders does there exist an idempotent-free exnq?

Minimal nonassociativity for loops

An open problem

Let Q be a loop of order n. There are $3n^2 - 3n + 1$ triples (x, y, z) such that $1 \in \{x, y, z\}$. Each of them is associative. Does there exist a loop of order $n > 1$ with exactly $3n^2 - 3n + 1$ associative triples?

A related problem for involutory loops

A loop Q is *involutory* if $x^2 = 1$ for all $x \in Q$. Involutory loops may be obtained by a prolongation of idempotent quasigroups. In an involutory loop $x^2 \cdot x = 1 \cdot x = x \cdot 1 = x \cdot x^2$. The number of associative triples is at least $3n^2 - 3n + 1 + (n - 1) = 3n^2 - 2n$. Does there exist a involutory loop of order $n > 1$ with exactly $3n^2 - 2n$ associative triples?

A partial answer relating to involutory loops

No such loop for orders $n \leq 9$.

$\exists$ if $n - 1 = p \geq 13$, p a prime, or $n - 1 = q^2$, q odd and prime power.

Minimal associativity in abelian groups 1

Defining a parameter $u(G)$, $(G,+)$ an abelian group

$u(G)$ is the minimum size of $\{(x,y,z) \in G^3;\ \lambda(x)+\rho(y)+\mu(z)=0\}$, which is counted over all transformation λ, ρ and μ that have the property that $\lambda(x)+\rho(x)+\mu(x)=0$ for all $x \in G$.

Connecting $u(G)$ to the associativity index

Claim: If Q is an isotope of G, then $a(Q) \geq u(G)$.

Source: D & Valent: Designs, Codes and Cryptography **86** (2018).

Expressing $u(G)$ as $\min v((q_{ij}))$.

Here $S=(q_{ij})$ is a square matrix of non-negative integers indexed by elements of G, $\sum q_{ij} = |G|$, $v(S) = \sum_{\substack{i,j,k\in G\\ i+j+k=0}} a_i b_j c_k$, where

$$a_i = \sum_{i\in G} q_{ij}, \quad b_j = \sum_{j\in G} q_{ij} \ \text{ and } \ c_k = \sum_{\substack{k\in G\\ i+j+k=0}} q_{ij}.$$

Minimal associativity in abelian groups 2

Conjecture for finite quasigroups Q isotopic to abelian groups

There exists $\lambda > 0$ such that $a(Q) > \lambda|Q|^2$.
A stronger but perhaps more accessible is this problem:
Does there exist $\lambda > 0$ such that $u(G) > \lambda|G|^2$ for every finite abelian group G?

A much weaker result

For each $\varepsilon > 0$ there exists $n_0 > 0$ such that for G a finite abelian group $|G| > n_0 \;\Rightarrow\; u(G) > (3-\varepsilon)|G|$. Possible choices: $\varepsilon = 1/2$ and $n_0 = 30$.

Notation and a consequence

For Q a quasigroup and for α, β permutations of Q denote by $Q_{\alpha,\beta}$ the principal isotope with operation $x * y = \alpha(x)\beta(y)$.
We have: If G is abelian group, then $a(G_{\alpha,\beta)}) \geq (3-\varepsilon)|G|$.
An isotope of G is thus never an exnq. This is also true if G is a noncommutative group. However, that case is even less understood.

Associative triples in groups 1

Associative index in a principal isotope $Q_{\alpha,\beta}$.

$a(Q_{\alpha,\beta}) = |\{(x,y,z) \in Q^3;\ x\beta(\alpha(y)z) = \alpha(x\beta(y))z\}|.$

Associative index in a principal isotope $G_{\alpha,\beta}$.

$a(G_{\alpha,\beta}) = |\{(x,y,z) \in G^3;\ \rho(z)\alpha(y) = \beta(y)\lambda(x)\}|$, where
$\lambda(x) = x^{-1}\alpha(x)$ and $\rho(z) = \beta(z)z^{-1}$.
Proof: $\beta(\alpha(y)z)z^{-1}(\alpha(y))^{-1}\alpha(y) = \beta(y)(x\beta(y))^{-1}\alpha(x\beta(y))$ is the equality above. It may be written as $\rho(\alpha(y)z)\alpha(y) = \beta(y)\lambda(x\beta(y))$.
(x,y,z) runs through $Q^3 \iff (x\beta(y), y, \alpha(y)(z))$ runs through Q^3.

Consequence for α left orthomorphism or β right orthomorphism

λ or ρ a permutation $\Rightarrow a(G_{\alpha,\beta}) = |G|^2$. *Proof:* Let λ permute G. For any choice of y and z there $\exists!\ x \in G$ such that $\rho(z)\alpha(y) = \beta(y)\lambda(x)$.

Associative triples in groups 2

The number of fixed point free translations

Easy to verify: Let $|G| = n$. In $G_{\alpha,\beta}$ there are $n - |\operatorname{Im}(\rho)|$ fixed point free left translations, and $n - |\operatorname{Im}(\lambda)|$ fixed point free right translations.
(Here, $\lambda(x) = x^{-1}\alpha(x)$ and $\rho(x) = \beta(x)x^{-1}$.)
In other words $\delta(G_{\alpha,\beta}) = 2n - |\operatorname{Im}(\rho)| - |\operatorname{Im}(\lambda)|$.
If $G_{\alpha,\beta}$ is extremely nonassociative, then ρ and λ are permutations, since $a(Q) \geq 2|Q| - |\mathrm{I}(Q)| + \delta(Q)$, for any quasigroup Q.
However, if λ or ρ is a permutation, then $a(G_{\alpha,\beta}) = n^2$.
Hence: A quasigroup isotopic to a group is never extremely nonassociative.

Simplification for abelian groups

Write $\rho(z) + \alpha(y) = \beta(y) + \lambda(x)$ as $\rho(z) + \alpha(y) = \lambda(x) + \beta(y)$ and subtract y. We obtain $\rho(z) + \lambda(y) = \lambda(x) + \rho(y)$. Minimum $a(G_{\alpha,\beta})$ is equal to the minimum of solutions (x, y, z) when λ and ρ run through transformations that may be expressed as $\sigma - \mathrm{id}_G$.

Computational results

The least associative index for small values

n	2	3	4	5	6	7	8	9	10
$a(n)$	8	9	16	15	16	17	16	9	≥ 11

Surpluses for loops and involutory loops

Call $a(Q) - (3n^2 - 3n + 1)$ the *surplus* for loops, $|Q| = n$, and $a(Q) - (3n^2 - 2n)$ an *(involutory) surplus* if Q is an involutory loop.

n	2	3	4	5	6	7	8	9	10
general	1	8	27	13	13	20	17	16	≤ 11
involutory	–	–	24	24	20	21	25	28	0

Minimal associativity index $m(G)$ for isotopes of a group G

G	$\mathbb{Z}_5$	$\mathbb{Z}_6$	S_3	$\mathbb{Z}_7$	$\mathbb{Z}_8$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	D_8	Q_8	E_8
$m(G)$	20	26	28	40	48	48	48	48	64

Results on the average value of associativity index

Ingredients upon which the results are based

$a(Q) = \sum_{x,y\in Q} |\operatorname{Fix}([L_x, R_y])|$ - points fixed by translation commutators; and $\sum_{\varphi,\psi\in S_n} |\operatorname{Fix}([\varphi,\psi])| = n^3(n-1)!(n-2)!$. - an easy result based on Burnside's Lemma.

Average associativity index over all principal isotopes

$\frac{1}{(n!)^2} \sum_{\alpha,\beta\in\operatorname{Sym}(Q)} a(Q_{\alpha,\beta}) = n^2 \left(1 + \frac{1}{n-1}\right)$, whenever $|Q| = n$.
Hence: Average value of $a(Q)$, $|Q| = n$, is $n^2(1 + (n-1)^{-1})$.

Average associativity index over one sided principal isotopes

Q a quasigroup of order n, $\alpha \in \operatorname{Sym}(Q)$ fixed, $f_x = |\operatorname{Fix}(R_x\alpha)|$, $\forall\ x \in Q$.
$\frac{1}{n!} \sum_{\varphi\in\operatorname{Sym}(Q)} a(Q_{\alpha,\varphi}) = \frac{n}{n-1} \sum_{x\in Q}(f_x^2 - 2f_x + n) \geq n^2$.
Equality to $n^2 \iff \alpha^{-1}$ a (left) orthomorphism of Q ($\alpha(x)\backslash x$ permutes Q).

Papers that contain the reported results

- A. Drápal and V. Valent: *Few associative triples, isotopisms and groups,* Des. Codes Cryptogr. **86** (2018), 555–568.
- A. Drápal and V. Valent: *High nonassociativity in order 8 and an associative index estimate,* J. Combin. Des. **27** (2019), 205–228.
- A. Drápal and V. Valent: *Extreme nonassociativity in order nine and beyond,* J. Combin. Des. **28** (2020), 33–48.
- A. Drápal and P. Lisoněk: *Maximal nonassociativity via nearfields,* Finite Fields Appl. **62** (2020) 101610,
- A. Drápal and I. M. Wanless: *Maximally nonassociative quasigroups via quadratic orthomorphisms,* Alg. Comb. **4** (2021), 501–515.
- A. Drápal and J. Hora: *Nonassociative triples in involutory loops and in loops of small order,* Comment. Math. Univ. Carolin. **61** (2020), 459–479.

What made a complete search feasible

Estimate for the # of elementary associative triples

$|\mathrm{I}(Q)| - |Q| + S$, where for $Q = \{x_1, \dots, x_n\}$, $a_i = |e^{-1}(x_i)|$ and $b_i = |f^{-1}(x_i)|$, $S = \sum_{i=1}^{n} \big(a_i^2 + b_i^2 + a_i b_i\big) - \sum_{i=1}^{k} \big(a_i + b_i\big)$.

$|\mathrm{I}(Q)| - |Q| + S \geq 2|Q| - |\mathrm{I}(Q)| + \delta_L(Q) + \delta_R(Q)$,

$\delta_L(Q) = |\{i; a_i = 0\}| = |\{x \in Q; \mathrm{Fix}(L_x) = \emptyset\}|$ and $\delta_R(Q) = |\{i; b_i = 0\}|$.

The search may be parallelized by prefilling e and f. A partially filled Latin square is being completed bottom down (row by row) and left to right (cell by cell) until a nonelementary ass. triple is found. Such triples are *diagonal* (x, x, x) and *nondiagonal*. Search can be speeded by this fact:

At the time of a nonelemenatry nondiagonal associative triple only 1 constituent is missing.

The *time* of ass. triple (x, y, z) is the pair (a, b) such that with ab both $x(yz)$ and $(xy)z$ can be computed (by using only that part of the latin square that precedes (a, b)). Constituents: xy, $xy \cdot z$, $x \cdot yz$, yz.

The situation with two operations

$x * (y \circ z) = (x * y) \circ z$

Let $*$ and $\circ$ be two quasigroup operations upon a set Q. Define $a_2(*, \circ)$ to be the number of all $(x, y, z) \in Q^3$ such that $x * (y \circ z) = (x * y) \circ z$.

Expressing by translations

Denote by L and R the translations of $(Q, *)$, and by λ and ρ the translations of $(Q, *)$. Then $a_2(*, \circ) = \sum_{x,z} |\operatorname{Fix}([L_x, \rho_z])|$. The right translations of $*$ and left translations of $\circ$ are not involved.

Average values

Denote by $*_{\alpha,\beta}$ the operation of the principal isotope. Thus $x *_{\alpha,\beta} y = \alpha(x) * \beta(y)$. $\frac{1}{(n!)^4} \sum_{\alpha,\beta,\gamma,\delta} a_2(*_{\alpha,\beta}, \circ_{\gamma,\delta}) = n^2(1 + \frac{1}{n-1})$.
The same average value as in one-operation case. In fact, $a_2(*_{\alpha,\beta}, \circ_{\gamma,\delta}) = a_2(*_{\sigma,\beta}, \circ_{\gamma,\tau})$, so for the computation only β and γ are relevant.

Minimum associative triples in two operations

$a_2(n) =$ minimum $a_2(*, \circ)$ for order n

Presently $a_2(n)$ known only up to $n = 5$. Comparison:

n	2	3	4	5
$a(n)$	8	9	16	15
$a_2(n)$	8	9	8	9

Spectrum in order 5

2 op: 9, 11, ..., 63, 65, 67, 68, 69, 71, 74, 76, 77, 79, 80, 89, 125
1 op: 15,..., 57, 59, 62, 63, 74, 79, 80, 89, 125

A problem

Do there exist quasigroups $(Q, *)$ and $(Q, \circ)$ of order $n > 1$ such that both are isotopic to a group and $a_2(*, \circ) = n$?