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The second Yang–Baxter homology for the HOMFLYPT polynomial

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ABSTRACT

In this paper, we adjust the Yang–Baxter operators constructed by Jones for the HOMFLYPT polynomial. Then we compute the second homology for this family of Yang–Baxter operators. It has the potential to yield 2-cocycle invariants for links.

Keywords: Knots and links; quandle; rack; Yang–Baxter equation; Jones polynomial; HOMFLYPT polynomial; 2-cocycles.

Mathematics Subject Classification 2020: Primary: 57K18; Secondary: 16T25

1. Introduction

The Yang–Baxter equation[a] was first introduced independently in a study of many body quantum system by Yang [18] and statistical mechanics by Baxter [1]. Since the discovery of the Jones polynomial [8] in 1984, solutions to the Yang–Baxter equation have become important for knot theory. In particular, Jones [8] and Turaev [16] built a machinery to construct link invariants using Yang–Baxter operators and the family of Yang–Baxter operators from the representation of A^1 series lead to sl_m polynomial invariants whose "limit" is the HOMFLYPT polynomial [7, 14]. Racks and quandles give special examples of Yang–Baxter operators. Homology theory

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[a]The name Yang–Baxter equation was coined by Ludvig Faddeev.

of racks and quandles were introduced in [3, 6]. Carter, Elhamdadi and Saito [2] defined a (co)homology theory for set-theoretic Yang–Baxter operators generalizing this homology. Homological and homotopical invariants of links using set-theoretic Yang–Baxter operators were studied in [2–4, 11, 12, 19]. The homology theory of general Yang–Baxter operators were developed by Lebed [10] and Przytycki [13] independently and this homology theory is equivalent to the one defined in [2] when restricted to the set-theoretic Yang–Baxter operators [15]. In the first part of this paper, we give a detailed proof that after modifying the Yang–Baxter matrix obtained from A^1 series to be column unital, they are still Yang–Baxter operators. Furthermore, this new family of operators also lead to the sl_m polynomial invariants [17], which is implicit in [8]. The homology can be defined for any column unital Yang–Baxter operators, see [13]. In the second part of the paper, we compute the second homology of the column unital Yang–Baxter operators corresponding to sl_m link invariants denoted by $R_{(m)}$ (see Theorem 3.3).

2. Column Unital Yang–Baxter Operators

Inspired by statistical mechanics, Jones constructed the Yang–Baxter operators leading to the Jones and HOMFLYPT polynomials, see [9, 16] for more information on the use of Yang–Baxter operators in knot theory.

Definition 2.1. Let k be a commutative ring and V be a k-module. If a k-linear map, $R : V \otimes V \to V \otimes V$, satisfies the following equation called the Yang–Baxter equation:

$$(R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R),$$

then we say R is a pre-Yang–Baxter operator. If, in addition, R is invertible, then we say R is a Yang–Baxter operator.

Definition 2.2. Let $k = \mathbb{Z}[q, q^{-1}]$, m be a positive integer, and V_m be the free k-module generated by the set $X_m = \{v_1, \ldots, v_m\}$ with ordering $v_a \leq v_b$ if and only if $a \leq b$. Express a k-linear operator $R : V_m \otimes V_m \to V_m \otimes V_m$ by $R(v_a \otimes v_b) = \sum_{1 \leq c,d \leq m} R^{ab}_{cd} v_c \otimes v_d$. Jones' Yang–Baxter operator on level m is given by the following coefficients:

$$R^{ab}_{cd} = \begin{cases} -q & \text{if } a = b = c = d; \\ 1 & \text{if } d = a \neq b = c; \\ q^{-1} - q & \text{if } c = a < b = d; \\ 0 & \text{otherwise,} \end{cases}$$

where a, b, c, d are integers satisfying $1 \leq a, b, c, d \leq m$.

In this section, we give a detailed proof that the family of column unital operators defined in Theorem 2.3 are Yang–Baxter operators. These operators are obtained from the Jones' Yang–Baxter operators by dividing each column by the sum of elements in the column and substitution $y^2 = \frac{1}{1+q^{-1}-q}$.

Theorem 2.3. *Let $k = \mathbb{Z}[y, y^{-1}]$, m be a positive integer, and V_m be the free k-module generated by the set $X_m = \{v_1, \dots, v_m\}$ with ordering $v_a \leq v_b$ if and only if $a \leq b$. Then the k-linear operator $R_{(m)} : V_m \otimes V_m \to V_m \otimes V_m$ given by the coefficients*

$$R^{ab}_{cd} = \begin{cases} 1 & \text{if } d = a \geq b = c; \\ y^2 & \text{if } d = a < b = c; \\ 1 - y^2 & \text{if } c = a < b = d; \\ 0 & \text{otherwise,} \end{cases}$$

where a, b, c, d are integers satisfying $1 \leq a, b, c, d \leq m$, is a Yang–Baxter operator for each $m \geq 1$.

It follows directly that the inverse of these operators is

$$(R^{-1})^{ab}_{cd} = \begin{cases} 1 & \text{if } d = a \leq b = c; \\ y^{-2} & \text{if } d = a > b = c; \\ 1 - y^{-2} & \text{if } c = a > b = d; \\ 0 & \text{otherwise,} \end{cases}$$

where a, b, c, d are integers satisfying $1 \leq a, b, c, d \leq m$.

Before the proof of Theorem 2.3, we set up some notations and give Proposition 2.4. Throughout the paper, we will write R, V, X for $R_{(m)}, V_{(m)}, X_{(m)}$ defined in Theorem 2.3, respectively. In any statement, whenever we use R, V, X, it implies the statement is true for $R_{(m)}, V_{(m)}, X_{(m)}$, $\forall\, m = 2, 3, \dots$. We will use integers $1 \leq a, b, c \leq m$ to represent the basis elements v_a, v_b, v_c and (a, b, c) for the tensor product $v_a \otimes v_b \otimes v_c$.

Proposition 2.4. *$R(a, a) = (a, a)$ agrees with the formulas $R(a, b) = (1 - y^2)(a, b) + y^2(b, a)$ when $a < b$, $R(a, b) = (b, a)$ when $a > b$ by substituting $b = a$.*

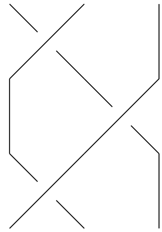
Proof. $R(a, a) = (a, a) = (1 - y^2)(a, a) + y^2(a, a)$. □

Now, we prove Theorem 2.3.

Proof. For $m = 1$, the Yang–Baxter equation hold trivially.

We consider the cases of $m \geq 2$.

Let $a \leq b \leq c$ for $a, b, c \in X_{(m)}$, then by Proposition 2.4, we need to check in total six cases for the Yang–Baxter equation, which are (a, b, c); (b, a, c); (a, c, b); (b, c, a); (c, a, b); $(c, b, a) \in X^3_{(m)}$. We start from the case of (a, b, c). From the left-hand side of the Yang–Baxter equation, computing terms by terms, we get the



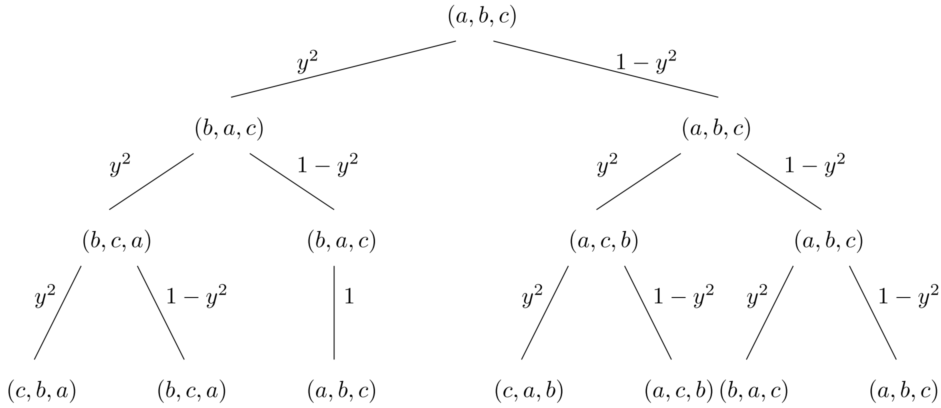


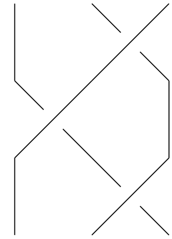
Fig. 1. Computational tree for the left-hand side of the Yang–Baxter equation of (a, b, c).

following (see Fig. 1):

$$
\begin{aligned}
&(R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V)(a, b, c)\\
&\qquad = (R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R)(y^2(b, a, c) + (1 - y^2)(a, b, c)),\\
&(R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R)(y^2(b, a, c))\\
&\qquad = (R \otimes \mathrm{Id}_V)(y^2y^2(b, c, a) + (1 - y^2)y^2(b, a, c)),\\
&((R \otimes \mathrm{Id}_V)(y^2y^2(b, c, a)) = y^2y^2y^2(c, b, a) + (1 - y^2)y^2y^2(b, c, a),\\
&((R \otimes \mathrm{Id}_V)((1 - y^2)y^2(b, a, c)) = (1 - y^2)y^2(a, b, c)
\end{aligned}
$$

and

$$
\begin{aligned}
&(R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R)((1 - y^2)(a, b, c))\\
&\qquad = (R \otimes \mathrm{Id}_V)(y^2(1 - y^2)(a, c, b) + (1 - y^2)(1 - y^2)(a, b, c)),\\
&(R \otimes \mathrm{Id}_V)(y^2(1 - y^2)(a, c, b))\\
&\qquad = y^2y^2(1 - y^2)(c, a, b) + (1 - y^2)y^2(1 - y^2)(a, c, b),
\end{aligned}
$$



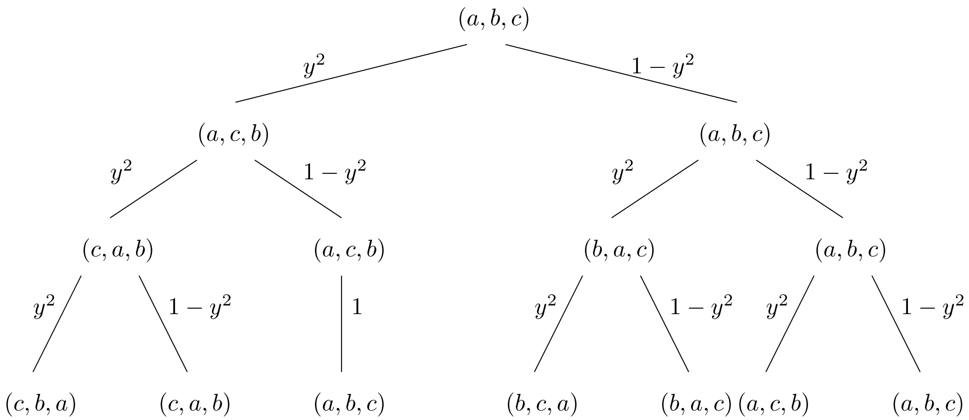


Fig. 2. Computational tree for the right-hand side of the Yang–Baxter equation of (a, b, c).

$$\begin{aligned}&(R \otimes \mathrm{Id}_V)((1-y^2)(1-y^2)(a,b,c))\\&\quad = y^2(1-y^2)(1-y^2)(b,a,c) + (1-y^2)(1-y^2)(1-y^2)(a,b,c).\end{aligned}$$

Similarly, we deal with the right-hand side of the Yang–Baxter equation, computing terms by terms, we get the following (see Fig. 2):

$(\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes R)(a,b,c) = (\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V)(y^2(a,c,b) + (1-y^2)(a,b,c))$,

$$\begin{aligned}(\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V)(y^2(a,c,b)) &= \mathrm{Id}_V \otimes R)(y^2y^2(c,a,b) + (1-y^2)y^2(a,c,b)),\\(\mathrm{Id}_V \otimes R)(y^2y^2(c,a,b)) &= y^2y^2y^2(c,b,a) + (1-y^2)(1-y^2)y^2(c,a,b),\\(\mathrm{Id}_V \otimes R)((1-y^2)y^2(a,c,b)) &= (1-y^2)y^2(a,b,c)\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{Id}_V \otimes R) \circ (R \otimes \mathrm{Id}_V)((1-y^2)(a,b,c))\\&\quad = \mathrm{Id}_V \otimes R)(y^2(1-y^2)(b,a,c) + (1-y^2)(1-y^2)(a,b,c)),\end{aligned}$$

$$\begin{aligned}&\mathrm{Id}_V \otimes R)(y^2(1-y^2)(b,a,c))\\&\quad = y^2y^2(1-y^2)(b,c,a) + (1-y^2)y^2(1-y^2)(b,a,c),\\&\mathrm{Id}_V \otimes R)((1-y^2)(1-y^2)(a,b,c))\\&\quad = y^2(1-y^2)(1-y^2)(a,c,b) + (1-y^2)(1-y^2)(1-y^2)(a,b,c).\end{aligned}$$

Both expressions are equal, thus prove Yang–Baxter equation holds for (a,b,c). The other cases can be checked directly in a similar way. □

Remark 2.5. From our proof and Figs. 1 and 2, we can conclude more:

$$\begin{aligned}&(R\otimes \mathrm{Id}_V)\circ(\mathrm{Id}_V\otimes R)\circ(R\otimes \mathrm{Id}_V)(a,b,c)\\&\quad = [[y^2y^2y^2(c,b,a) + (1-y^2)y^2y^2(b,c,a)] + [(1-y^2)y^2(a,b,c)]]\\&\qquad + [[y^2y^2(1-y^2)(c,a,b) + (1-y^2)y^2(1-y^2)(a,c,b)]\\&\qquad + [y^2(1-y^2)(1-y^2)(b,a,c) + (1-y^2)(1-y^2)(1-y^2)(a,b,c)]],\\&(\mathrm{Id}_V\otimes R)\circ(R\otimes \mathrm{Id}_V)\circ(\mathrm{Id}_V\otimes R)(a,b,c)\\&\quad = [[y^2y^2y^2(c,b,a) + (1-y^2)(1-y^2)y^2(c,a,b)] + [(1-y^2)y^2(a,b,c)]]\\&\qquad + [[y^2y^2(1-y^2)(b,c,a) + (1-y^2)y^2(1-y^2)(b,a,c)]\\&\qquad + [y^2(1-y^2)(1-y^2)(a,c,b) + (1-y^2)(1-y^2)(1-y^2)(a,b,c)]].\end{aligned}$$

Terms in the sum correspond to the leaves of the computational tree. Square brackets group terms according the structure of the tree (see Figs. 1 and 2).

Important observation is that if we transform the result of the left-hand side of (a,b,c) by first switching the position of a and c and then reversing the order of the triple, we obtain exactly the result of right-hand side of (a,b,c) square bracket-wisely. This observation can actually reduce the number of cases to check, which is important for us to compute the higher level homology in the future.

As mentioned before, the family of Yang–Baxter operators, $R_{(m)}$, has the property that summation of elements in each column of the matrix presentation equals to 1. They are obtained from the Yang–Baxter operators leading to the Jones and HOMFLYPT polynomials [8, 16] by normalizing each column. However, normalizing columns of Yang–Baxter operators do not always produce Yang–Baxter operators in general.

Counterexample 2.6. The following Yang–Baxter operator leading to the Kauffman two-variable polynomial (see [16] for details) with substitution $m = 4$, $\nu = -1$

is a counterexample:

$$\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & q-q^{-1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q-q^{-1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & q-2q^{-1}+(q^{-3}) & 0 & 0 & (q^{-2}-1) & 0 & 0 & (q^{-2}-1) & 0 & 0 & q^{-1} & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & (q^{-2}-1) & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q-q^{-1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & (q^{-2}-1) & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-q^{-1} & 0 & 0 & 1 & 0\\
0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{pmatrix}$$

This matrix as a k-linear operator from $V \otimes V$ to $V \otimes V$, with $k = \mathbb{Z}[q, q^{-1}]$ and $V = k\{v_1, v_2, v_3, v_4\}$ the free k-module generated by four elements, is a Yang–Baxter operator with the standard basis in tensor product of $V \otimes V$. However, if we divide the elements of each column by the summation of the elements in the corresponding column, it is no longer a Yang–Baxter operator. We have checked this by using Mathematica directly.

3. Computation of Homology for Yang–Baxter Operators Leading to HOMFLYPT Polynomial

In this section, we are interested in the second homology of $R_{(m)}$. First, we recall the definition of Yang–Baxter homology for column unital operators. Let k be a commutative ring, $V = k\mathrm{X}$ be the free k-module generated by basis in X, and let the chain modules be $C_n(R) = V^{\otimes n}$. The boundary homomorphism $\partial_n : C_n(R) \to C_{n-1}(R)$ is given as follows:

$$\partial_n = \sum_{i=1}^{n} (-1)^i (d^l_{i,n} - d^r_{i,n}).$$

The face maps $d^l_{i,n}$ and $d^r_{i,n}$ are illustrated in Fig. 3, where going from top to bottom, and whenever we meet a crossing we apply the Yang–Baxter operator R and we delete the first tensor factor or the last tensor factor at the bottom for $d^l_{i,n}$ and $d^r_{i,n}$, respectively. See [13, 15] for details.

Consider pre-Yang–Baxter operators $R : V \otimes V \to V \otimes V$ given on the basis X^2 by

$$R(a, b) = \sum_{c,d} R^{a,b}_{c,d} \cdot (c, d)$$

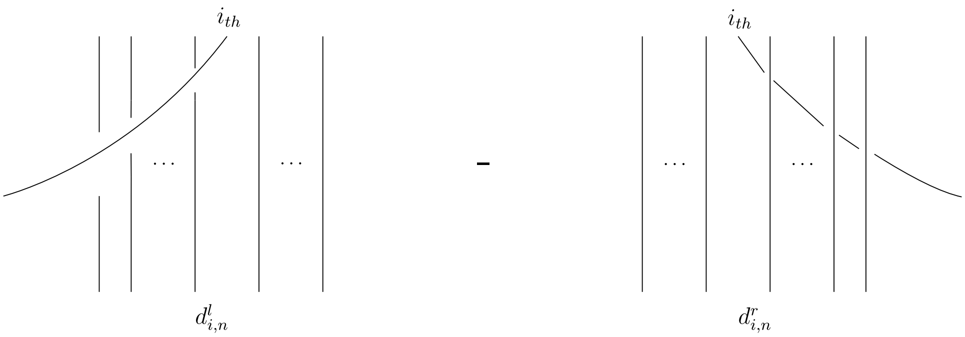


Fig. 3. Face maps.

with column unital condition, that is $\sum_{c,d} R^{a,b}_{c,d} = 1$ for every $(a,b) \in X^2$. Now $C_1(R) = V$ and $C_2(R) = V^{\otimes 2}$ and

$$\begin{aligned}\partial_2(a,b) &= \sum_{i=1}^{2}(-1)^i(d^\ell_i(a,b) - d^r_i(a,b))\\ &= (b) - \sum_{c,d} R^{a,b}_{c,d}(d) - \left(\sum_{c,d} R^{a,b}_{c,d}(c) - (a)\right)\\ &= (a) + (b) - \sum_{c,d} R^{a,b}_{c,d}((c)+(d))\end{aligned}$$

and

$$\partial_3(a,b,c) = \sum_{i=1}^{3}(-1)^i(d^\ell_i(a,b,c) - d^r_i(a,b,c)).$$

Now we go back to analysis of the chain complex for the column unital matrices $R_{(m)}$ in Theorem 2.3. Recall that

$$R^{ab}_{cd} = \begin{cases} 1 & \text{if } d = a \geq b = c;\\ y^2 & \text{if } d = a < b = c;\\ 1 - y^2 & \text{if } c = a < b = d;\\ 0 & \text{otherwise,}\end{cases}$$

In particular, for $m = 2$ we have the matrix

$$R_{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1-y^2 & 1 & 0\\ 0 & y^2 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad R^{-1}_{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & y^{-2} & 0\\ 0 & 1 & 1-y^{-2} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

In Lemma 3.1, we prove that the second boundary map is trivial.

Lemma 3.1. *For the family of column unital Yang–Baxter operators in Theorem* 2.3, $\partial_2 = 0$ *and*

$$H_1(R) = C_1(R_{(m)}) = V$$

and

$$\ker\partial_2 = C_2(R_{(m)}) = V^{\otimes 2}.$$

Proof. We check now that $\partial_2(a,b) = 0$ for any pair $a, b \in X^2$. The main reason for $\partial_2 = 0$ is that if $\{a,b\} \neq \{c,d\}$ then $R^{a,b}_{c,d} = 0$ and the column unital property. That is for $a, b \in X$:

$$\sum_{c,d} R^{a,b}_{c,d}(c,d) = R^{a,b}_{a,b}(a,b) + R^{a,b}_{b,a}(b,a) \text{ with } R^{a,b}_{a,b} + R^{a,b}_{b,a} = 1,$$

so $\partial_2(a,b) = (a) + (b) - (R^{a,b}_{a,b} + R^{a,b}_{b,a})((a) + (b)) = 0$.

Thus

$$H_1(R_{(m)}) = C_1(R_{(m)}) = V$$

and

$$\ker\partial_2 = C_2(R_{(m)}) = V^2. \qquad \square$$

To compute $H_2(R_{(m)})$, we need to understand $im\partial_3$. The following lemma will be used later in computation.

Lemma 3.2. *For the column unital Yang–Baxter operators in Theorem* 2.3, *we have*

(1) $\partial_3(v_m, a_1, a_2) = 0$ *and* $\partial_3(a_1, a_2, v_1) = 0$ *for all* $a_1, a_2 \in X$, *where as before* v_m *is the largest element and* v_1 *is the smallest element in* X;
(2) $\partial_3(a_1, a_2, a_3) = 0$ *if either* $a_1 \geq a_i$ *for all* $i = 1, 2, 3$ *or* $a_3 \leq a_j$ *for all* $j = 1, 2, 3$, *for all* $a_1, a_2, a_3 \in X$.

Proof. Part (1) follows from Lemma 3.1. $\partial_3(v_m, a_1, a_2) = [d^l_1 - d^r_1](v_m, a_1, a_2) - v_m \otimes \partial_2(a_1, a_2)$, by Lemma 3.1, $\partial_3(v_m, a_1, a_2) = [d^l_1 - d^r_1](v_m, a_1, a_2)$. Note that $R(a_1, a_2) = (a_2, a_1)$ whenever $a_1 \geq a_2$, $\partial_3(v_m, a_1, a_2) = [d^l_1 - d^r_1](v_m, a_1, a_2) = (a_1, a_2) - (a_1, a_2) = 0$. Similarly, $\partial_3(a_1, a_2, v_1) = 0$.

Part (2) follows from part (1) by considering the subchain complex given by the subspace $\{v_1, v_2, \dots, a_1\}$ of V_m or $\{a_3 \dots, a_{m-1}, a_m\}$ of V_m, respectively. $\square$

The main result of this section is as follows. Notice that the ring k can be either $\mathbb{Z}[y^{\pm}]$ or $\mathbb{Z}[y]$.

Theorem 3.3. *Let R be a unital Yang–Baxter operator giving HOMFLYPT polynomial on level m in Theorem* 2.3, *then*

$$H_2(R) = k^{1+\binom{m}{2}} \oplus (k/(1-y^2))^{\binom{m}{2}} \oplus \Big(k/(1-y^4)\Big)^{m-1}.$$

Proof. First, we compute ∂_3. Let $a < b < c$, we need to consider 13 cases, which are

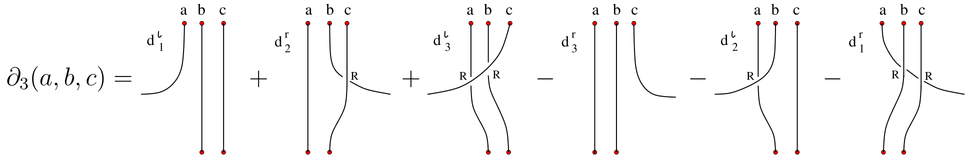
$$(a,b,c);\ (b,c,a);\ (c,a,b);\ (b,a,c);\ (a,c,b);\ (c,b,a);\ (a,a,b);\ (b,b,a);$$
$$(a,b,b);\ (b,a,a);\ (a,b,a);\ (b,a,b);\ (a,a,a)$$

By Lemma 3.2, we have $\partial_3(b,c,a) = \partial_3(c,a,b) = \partial_3(c,b,a) = \partial_3(b,b,a) = \partial_3(b,a,a) = \partial_3(a,b,a) = \partial_3(b,a,b) = \partial_3(a,a,a) = 0$.

∂_3 provides non-trivial relations in the homology for the 5 remaining cases (however, they are not all linearly independent). Let us demonstrate the calculation of $\partial_3(a,b,c)$.

We make calculation easy by considering graphical interpretation of face maps d_i^ϵ, starting from the defining formula

$$\partial_3 = d_1^\ell + d_2^r + d_3^\ell - (d_3^r + d_2^\ell + d_1^r).$$



From these diagrams we compute (keeping the terms in the same order as in the figure):

$$\begin{aligned}\partial_3(a,b,c) &= (b,c) + (y^2(a,c) + (1-y^2)(a,b)) \\ &\quad + (y^4(a,b) + y^2(1-y^2)(c,b) + (1-y^2)y^2(a,c) + (1-y^2)^2(b,c)) \\ &\quad - (a,b) - (y^2(a,c) + (1-y^2)(b,c)) \\ &\quad - (y^4(b,c) + y^2(1-y^2)(b,a) + (1-y^2)y^2(a,c) + (1-y^2)^2(a,b)) \\ &= (1-y^2)((b,c) - (a,b) + y^2((c,b) - (b,a))).\end{aligned}$$

The longest calculations are those of d_3^ℓ and d_1^r. In the next picture we illustrate how to compute quickly d_3^ℓ:

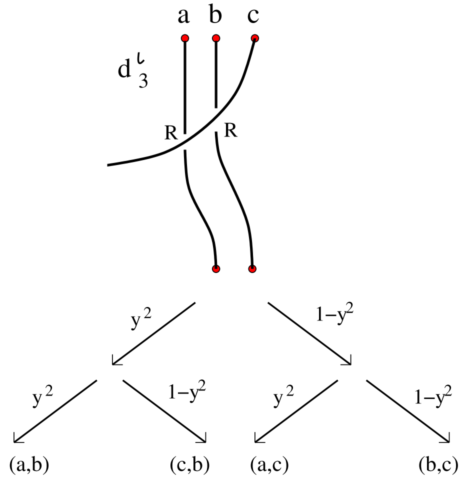


Figure 3 $d_3^\ell = y^4(a,b) + y^2(1-y^2)(c,b) + (1-y^2)y^2(a,c) + (1-y^2)^2(b,c)$.

The computation for d_1^r is similar. We can also use the symmetry that is a and c switch roles and (x,y) goes to (y,x). Thus we get $d_1^r = y^4(b,c) + y^2(1-y^2)(b,a) + (1-y^2)y^2(a,c) + (1-y^2)^2(a,b)$.

With some efforts, we get the following non-trivial differentials of elements with three distinct letters:

$$\partial_3(a,b,c) = (1-y^2)((b,c)-(a,b)+y^2((c,b)-(b,a))),$$
$$\partial_3(a,c,b) = (1-y^2)((b,c)-(a,c)+y^2((c,b)-(c,a))),$$
$$\partial_3(b,a,c) = (1-y^2)((a,c)-(a,b)+y^2((c,a)-(b,a))).$$

They are not independent as:

$$\partial_3(a,b,c) - \partial_3(a,c,b) - \partial_3(b,a,c) = 0.$$

Also, by Proposition 2.4, we have

$$\partial_3(a,a,b) = (1-y^2)((a,b)-(a,a)+y^2((b,a)-(a,a))),$$
$$\partial_3(a,b,b) = (1-y^2)((b,b)-(a,b)+y^2((b,b)-(b,a))).$$

From the following two equations, we see that the relations given by ∂_3 are generated by the images of (a,a,b) and (a,b,b) as follows:

$$\partial_3(a,b,c) = (1-y^2)((b,c)-(a,b)+y^2((c,b)-(b,a))) = \partial_3(b,b,c) + \partial_3(a,b,b)$$

and

$$\partial_3(b,a,c) = (1-y^2)((a,c)-(a,b)+y^2((c,a)-(b,a))) = \partial_3(a,a,c) - \partial_3(a,a,b).$$

Let us summarize the structure of the image $\partial_3(C_3)$. It is generated by

$$\partial_3(v_i,v_i,v_j) = (1-y^2)\big((v_i,v_j)-(v_i,v_i)+y^2((v_j,v_i)-(v_i,v_i)\big) \text{ for } i<j$$

and

$$\partial_3((v_i, v_i, v_j) + (v_i, v_j, v_j)) = (1 - y^4)((v_j, v_j) - (v_i, v_i)) \text{ for } i < j.$$

We notice quickly that $\partial_3((v_i, v_i, v_j) + (v_i, v_j, v_j))$ is generated by $m - 1$ elements $(v_j, v_j) - (v_1, v_1)$ with $m \geq j > 1$.

Consider the following new basis of kX^2 consisting of three groups of basis elements:

$X_0 = \{(v_1, v_1), (v_j, v_i) \text{ for } i < j\}$ that is $\binom{m}{2} + 1$ elements,

$X_1 = \{(v_i, v_j) - (v_i, v_i) + y^2((v_j, v_i) - (v_i, v_i)) \text{ for } i < j\}$ that is $\binom{m}{2}$ elements,

$X_2 = \{(v_j, v_j) - (v_1, v_1) \text{ with } m \geq j > 1\}$ that is $m - 1$ elements.

Clearly, $X_0 \sqcup X_1 \sqcup X_2$ form a basis of kX^2.

We look now at relations: in our basis, the matrix of relations is diagonal with 0 for elements in X_0, $(1 - y^2)$ for elements in X_1 and $(1 - y^4)$ for elements in X_2. Thus not only we proved that

$$H_2(R) = k^{1+\binom{m}{2}} \oplus (k/(1 - y^2))^{\binom{m}{2}} \oplus (k/(1 - y^4))^{m-1},$$

but we also found a basis of $C_2 = kX^2$ realizing the decomposition into cyclic submodules. □

From Theorem 3.3, we can easily see the rank of $\ker\partial_3(R_{(m)})$.

Corollary 3.4. $\mathrm{Rank}(\ker\partial_3(R_{(m)})) = \frac{(m+1)(2m^2-3m+2)}{2}$.

Proof. Rank of the kernal ∂_3 is the rank of C_3 minus the number of non-zero elements in the diagonal relation matrix of ∂_3, which is exactly the numbers of $(1 - y^2)$ and $(1 - y^4)$. Thus

$$\mathrm{Rank}(\ker\partial_3(R_{(m)})) = m^3 - \binom{m}{2} - (m - 1) = \frac{(m + 1)(2m^2 - 3m + 2)}{2}.$$ □

4. Further Computations and Future Work

Here we summarize all data obtained with the help of computer. Because of the limitation of the computation program, the computation were done over the ring $\mathbb{Q}[y]$. In [15], we formulated a conjecture about the homology of $R_{(m)}$ when $m = 2$ as follows.

Conjecture 4.1 ([15]). When $m = 2$, $H_n = k^2 \bigoplus (k/(1 - y^2))^{a_n} \bigoplus (k/(1 - y^4))^{s_{n-2}}$, where $s_n = \Sigma_{i=1}^{n+1} f_i$ is the partial sum of Fibonacci sequence, where $f_1 = 1 = f_2$ and a_n is given by $2^n = 2 + a_{n-1} + s_{n-3} + a_n + s_{n-2}$ with $a_1 = 0$.

Table 1. Second and third homology for small m.

H_n	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
H_2	(4, 3, 2)	(7, 6, 3)	(11, 10, 4)	(16, 15, 5)	(22, 21, 6)
H_3	(4, 12, 6)	(8, 35, 12)	(15, 76, 20)	(26, 140, 30)	(42, 232, 42)

This conjecture is verified upto $n \leq 11$ by computer. In the paper [5], there is a discussion of various aspects of this conjecture. More computation is shown in Table 1, where (x, y, z) represents decomposition into x copies of k, y copies of $k/(1-y^2)$ torsion and z copies of $k/(1-y^4)$ torsion.

From the first row of the table, we can see that the results match with that of the formula in Theorem 3.3. From the second row of the table, we conjecture the formula for H_3 as follows.

Conjecture 4.2. $H_3(R_{(m)}) = k^{\frac{m(8-3m+m^2)}{6}} \bigoplus (k/(1-y^2))^{\frac{(m^2-1)(5m-6)}{6}} \bigoplus (k/(1-y^4))^{m(m-1)}$

Remark 4.3. (1) The ranks in Conjecture 4.2 sum up to the rank of $\ker\partial_3$.

(2) The rank of $H_3(R_{(2)})$ in Conjecture 4.1 agrees with the rank of $H_3(R_{(2)})$ in Conjecture 4.2.

Computations and patterns observed so far suggest that there are only two types of torsion elements $k/(1-y^2)$ and $k/(1-y^4)$. However, this is only checked upto the strength of the computer program. By analyzing the boundary maps in general, we hope to gain more information about $H_n(R_{(m)})$. The first step towards this goal is the following observation.

Remark 4.4. The factor $(1-y^2)$ divides every element in $\mathrm{Im}(\partial_n)$. This follows from the fact that when setting $1-y^2=0$, $d_i^\ell = d_i^r$. Thus, we have $\partial_n(a_1, \ldots, a_n) \subset (1-y^2)V^n$, where $a_i \in X_m$, $i = 1, 2, \ldots, n$. One possible approach to compute $H_n(R_{(m)})$ is to decompose the boundary map along the factors $(1-y^2)^i$. In the first step, we ignore the branches with factor $(1-y^2)$ in the computational tree, see Fig. 3. Generally, in the ith step, we ignore the paths in the computational tree which are going i or more times to the right.

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References

[1] R. J. Baxter, Partition function of the eight-vertex lattice model, *Ann. Physics* **70** (1972) 193–228.

[2] J. S. Carter, M. Elhamdadi and M. Saito, Homology theory for the set-theoretic Yang–Baxter equation and knot invariants from generalizations of quandles, *Fund. Math.* **184** (2004) 3–54.

[3] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, Quandle Cohomology and State-Sum Invariants of Knotted Curves and Surfaces, *Trans. Amer. Math. Soc.* **355**(10) (2003) 3947–3989.

[4] J. Ceniceros, M. Elhamdadi, M. Green and S. Nelson, Augmented biracks and their homology, *Int. J. Math.* **25**(9) (2014) 1450087.

[5] M. Elhamdadi, M. Saito and E. Zappala, Skein theoretic approach to Yang–Baxter homology, preprint (2020), arXiv:2004.00691 [math.GT].

[6] R. Fenn, C. Rourke and B. Sanderson, An introduction to species and the rack space, in *Topics in Knot Theory* (*Proc. Topology Conference*, *Erzurum*), NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci., Vol. 399, ed. M. E. Bozhuyuk (Kluver Academic Publishers, 1993), pp. 33–35.

[7] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu, A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc.* **12**(2) (1985) 239–246.

[8] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (2) **126**(2) (1987) 335–388.

[9] V. F. R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math.* **137**(2) (1989) 311–334.

[10] V. Lebed, Homologies of algebraic structures via braidings and quantum shuffles, *J. Algebra* **391** (2013) 152–192.

[11] T. Nosaka, On homotopy groups of quandle spaces and the quandle homotopy invariant of links, *Topology Appl.* **158**(8) (2011) 996–1011.

[12] T. Nosaka, Quandle homotopy invariants of knotted surfaces, *Math. Z.* **274**(1–2) (2013) 341–365.

[13] J. H. Przytycki, Knots and distributive homology: From arc colorings to Yang–Baxter homology, in *New Ideas in Low Dimensional Topology*, Vol. 56 (World Scientific, 2015), pp. 413–488.

[14] J. H. Przytycki and P. Traczyk, Invariants of links of Conway type, *Kobe J. Math.* **4** (1987) 115–139.

[15] J. H. Przytycki and X. Wang, Equivalence of two definitions of set-theoretic Yang–Baxter homology and general Yang–Baxter homology, *J. Knot Theory Ramifications* **27**(7) (2018) 1841013.

[16] V. G. Turaev, The Yang–Baxter equation and invariants of links, *Invent. Math.* **92**(3) (1988) 527–553.

[17] X. Wang, Knot theory and algebraic structures motivated by and applied to knots, PhD thesis, George Washington University (2018).

[18] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967) 1312–1315.

[19] S. Y. Yang, Extended quandle spaces and shadow homotopy invariants of classical links, *J. Knot Theory Ramifications* **26**(3) (2017) 1741010.