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Equivalence of two definitions of set-theoretic Yang–Baxter homology and general Yang–Baxter homology

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ABSTRACT

In 2004, Carter, Elhamdadi and Saito defined a homology theory for set-theoretic Yang–Baxter operators (we will call it the "algebraic" version in this paper). In 2012, Przytycki defined another homology theory for pre-Yang–Baxter operators which has a nice graphic visualization (we will call it the "graphic" version in this paper). We show that they are equivalent. The "graphic" homology is also defined for pre-Yang–Baxter operators, and we give some examples of its one-term and two-term homologies. In the two-term case, we have found torsion in homology of Yang–Baxter operator that yields the Jones polynomial.

Keywords: Homology; pre-cubical module; pre-simplicial module; torsion; Yang–Baxter operators.

Mathematics Subject Classification 2010: 57M25, 18G60

1. Introduction

The Yang–Baxter equation was introduced independently by Yang(1967) [18] and Baxter (1972) [1]. It is well known that a certain solution of Yang–Baxter equation gives rise to the Jones polynomial [6]. In 2004, Carter, Elhamdadi and Saito defined a (co)homology theory for set-theoretic Yang–Baxter operators, from which they gave a way to generate link invariants, cocycle invariants [2]. In 2012, Przytycki gave a graphical definition of homology for a pre-Yang–Baxter operator [13]. We provide

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the definitions of two homology theories for set-theoretic Yang–Baxter operators in Sec. 2 and show their equivalence in Sec. 3. In Sec. 4, we give definitions of one-term and two-term homology of pre-Yang–Baxter operators, and show examples, in particular, we find torsion in two-term homology of Yang–Baxter operator that yields the Jones polynomial. In Sec. 5, we will mention some future direction. We start from basic definitions.

Definition 1.1. Let X be a set. If $R : X \times X \to X \times X$ is a function that satisfies

$$(R \times Id_X) \circ (Id_X \times R) \circ (R \times Id_X) = (Id_X \times R) \circ (R \times Id_X) \circ (Id_X \times R),$$

then we say R is a set-theoretic pre-Yang–Baxter operator and the equation above is a set-theoretic Yang–Baxter equation. If, in addition, R is invertible, then we say R is a set-theoretic Yang–Baxter operator.

Set-theoretic pre-Yang–Baxter operator leads to Yang–Baxter operator (see Definition 4.1) by putting $V = kX$, and extending $R : X \times X \to X \times X$ to $R : V \otimes V \to V \otimes V$. This Yang–Baxter operator is still called set-theoretic pre-Yang–Baxter operator.

2. Set-Theoretic Yang–Baxter Homology Theories

Given a set-theoretic pre-Yang–Baxter operator R, we have two approaches to homology built on R. The "algebraic" version defined in [2] and the "Graphic" version in [13]. We discuss them in the next two subsections. We prove the equivalence of them in Sec. 3.

We first review the "algebraic" version of set-theoretic Yang–Baxter homology theory based on [2] and then introduce the "graphic" version of set-theoretic Yang–Baxter homology theory.

2.1. "*Algebraic*" *homology of Carter*, *Elhamdadi*, *and Saito*

Definition 2.1. The set X together with a set-theoretic Yang–Baxter operator R, (X, R), is called in [2] a Yang–Baxter set. We represent a function R by $R(x_1, x_2) = (R_1(x_1, x_2), R_2(x_1, x_2))$.

We use the following notations. Let $\mathcal{I}_n$ be the n-dimensional cube I^n ($I = [0, 1]$) regarded as a CW (cubical) complex, where n is a positive integer.[a] Denote the k-skeleton by $\mathcal{I}_n^{(k)}$ with orientation given by the order of coordinate axes. In particular, every 2-face can be written as $\epsilon_1 \times \cdots \times \epsilon_{i-1} \times I_i \times \epsilon_{i+1} \times \cdots \times \epsilon_{j-1} \times I_j \times \epsilon_{j+1} \times \cdots \times \epsilon_n$, for some $1 \le i < j \le n$, where $\epsilon_k = 0$ or 1, and I_i, I_j denote two copies of I at the ith, jth positions, respectively.

[a]We deal here with a co-pre-cubic set (X_k, d^i_ϵ) where $X_k = I^k$ and co-face maps $d^{i,k-1}_\epsilon : X_{k-1} \to X_k$ defined by $d^i_\epsilon(x_1, x_2, \ldots, x_{k-1}) = (x_1, x_2, \ldots, x_{i-1}, \epsilon, x_i, \ldots, x_{k-1})$; they satisfy $d^j_\delta d^i_\epsilon = d^i_\epsilon d^{j-1}_\delta$ where $i < j$.

Definition 2.2. The Yang–Baxter coloring of $\mathcal{I}_n$ by a Yang–Baxter set (X,R) is a map $L: E(\mathcal{I}_n) \to X$, where $E(\mathcal{I}_n)$ denotes the set of edges of $\mathcal{I}_n$, with each edge oriented as above, such that if

$L(\epsilon_1 \times \cdots \times I_i \times \cdots \times 0_j \times \cdots \times \epsilon_n) = x,$
$L(\epsilon_1 \times \cdots \times 1_i \times \cdots \times I_j \times \cdots \times \epsilon_n) = y,$
then
$L(\epsilon_1 \times \cdots \times 0_i \times \cdots \times I_j \times \cdots \times \epsilon_n) = R_1(x,y),$
$L(\epsilon_1 \times \cdots \times I_i \times \cdots \times 1_j \times \cdots \times \epsilon_n) = R_2(x,y).$

Compare Fig. 1.

Definition 2.3. The initial path in $\mathcal{I}_n$ is the sequence of edges of $\mathcal{I}_n$,$(e_1,\ldots,e_n)$, where

$e_1 = I_1 \times 0_2 \times \cdots \times 0_n,$
$e_2 = 1_1 \times I_2 \times 0_3 \times \cdots \times 0_n,$
$\vdots$
$e_n = 1_1 \times 1_2 \cdots \times 1_{n-1} \times I_n.$

Lemma 2.4 ([2]). *Let (X,R) be a Yang–Baxter set, and $(e_1,\ldots,e_n)$ be the initial path of $\mathcal{I}_n$. For any n-tuple of elements of X, $(x_1,\ldots,x_n)$, there exists a unique Yang–Baxter coloring L of $\mathcal{I}_n$ by (X,R) such that $L(e_i) = x_i$ for all $i = 1,\ldots,n$.*

This lemma gives the following two properties:

- Each edge has the color uniquely induced by the n-tuple associated to the initial path of $\mathcal{I}_n$,
- Each k-face $\mathcal{J}$ of $\mathcal{I}_n$ has its induced initial path determined by the order of coordinates. Therefore, we can associate to it the k-tuple $(y_1,\ldots,y_k)$ determined by colors on its induced initial path. Denote this situation by $L(\mathcal{J}) = (y_1,\ldots,y_k)$. That is, $L(\mathcal{J})$ is the restriction of the function $L: E(\mathcal{I}_n) \to X$ of Definition 2.2 to the initial path of $\mathcal{J}$.

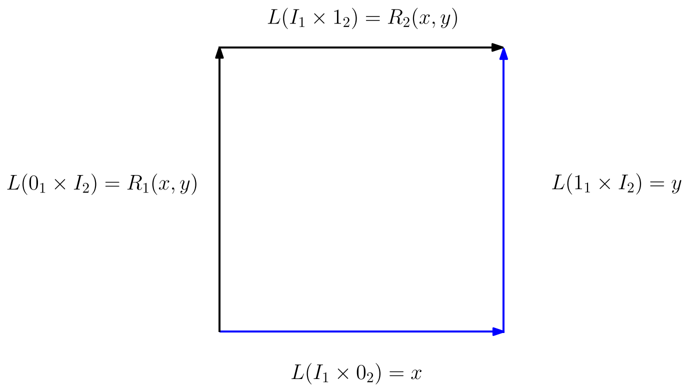

Fig. 1. Local behavior of Yang–Baxter coloring.

From these two facts, we have a way to map an n-tuple to $(n-1)$-tuple through the face maps in cubic homology theory.

Recall that ∂_n^C denotes the n-dimensional boundary map in the cubical homology theory. Thus $\partial_n^C(\mathcal{I}_n) = \sum_{i=1}^{2n} \epsilon_i \mathcal{J}_i$, where $\mathcal{J}_i$ is an $(n-1)$-face and $\epsilon_i = \pm 1$ depending on whether the orientation of $\mathcal{J}_i$ matches the induced orientation. For the induced orientation, we take the convention that the inward pointing normal to an $(n-1)$-face appears last in a sequence of vectors that specifies an orientation, and the orientation of the $(n-1)$-face is chosen so that this sequence agrees with the orientation of the n-cube.

Let (X, R) be a Yang–Baxter set. Let $C_n^{YB}(X)$ be the free Abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of X. Define a homomorphism ∂_n^A : $C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ by $\partial_n^A((x_1, \ldots, x_n)) = L(\partial_n^C(\mathcal{I}_n)) = \sum_{i=1}^{2n} \epsilon_i L(\mathcal{J}_i)$. We have $\partial_{n-1}^A \circ \partial_n^A = 0$, and $(C_*^{YB}(X), \partial_n^A)$ is a chain complex. As usual, we can define $H_n^A = ker\partial_n^A / im\partial_{n+1}^A$ to be the "algebraic" version of Yang–Baxter homology group [2].

2.2. *"Graphic" approach to Yang–Baxter homology*

In this homology theory, the chain groups are the same as before, that is $C_n^{YB}(X) = ZX^n$. We define the boundary homomorphism $\partial_n^G : C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ as follows, $\partial_n^G = \sum_{i=1}^{n} (-1)^i d_{i,n}$, where $d_{i,n} = d_{i,n}^l - d_{i,n}^r$. We can interpret the face maps through Fig. 2. The meaning of $d_{i,n}^l$, is illustrated in Fig. 3; $d_{i,n}^r$ can be described similarly.

We have an n-tuple as an input and each strand carries the corresponding element of the n-tuple. We track down the graph from top to bottom, and at each crossing we apply the fixed Yang–Baxter operator with input the ordered pair consists of two elements carried by the two strands right above the crossing (as in Fig. 4).

Then the left strand after the crossing carries the R_1 function value and the right strand after the crossing carries the R_2 function value. In the end, we ignore the element carried by the left most strand, and this procedure generates an $(n-1)$-tuple consisting of the $n-1$ elements carried by the other $n-1$ strands at the bottom.

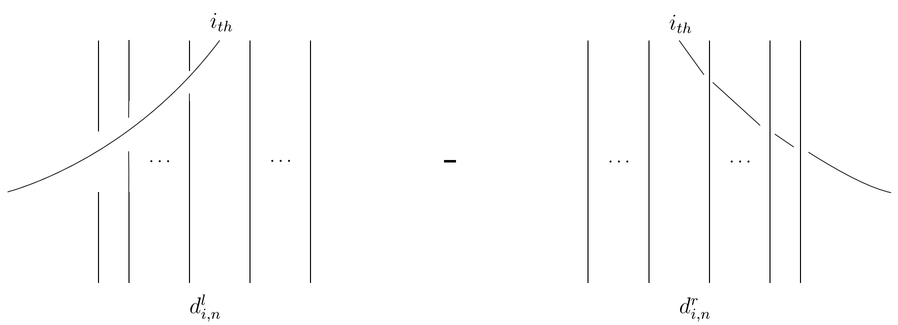

Fig. 2. A face map $d_{i,n}$.

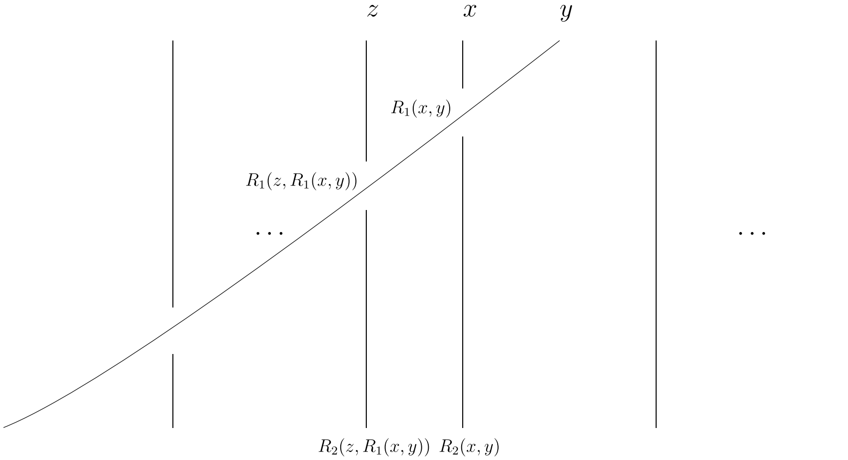

Fig. 3. A face map $d^l_{i,n}$.

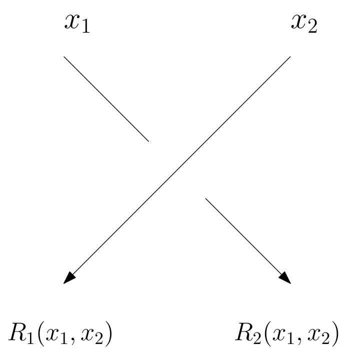

Fig. 4. Encoding R at each crossing.

One can easily check $(X^n, d^\epsilon_{i,n})$ form a pre-cubic set,[b] which implies that $(C^{YB}_*(X), \partial^G_n)$ is a chain complex. We define $H^G_n = ker\partial^G_n / im\partial^G_{n+1}$ as the "graphic" version of Yang–Baxter homology group.

[b]The pre-cubical set is a collection of sets X_n, $n \geq 0$, together with maps, called face maps or face operators,

$$d^\epsilon_i : X_n \to X_{n-1}, \quad 1 \leq i \leq n, \ \ \epsilon = 0, 1$$

such that:

$$d^\epsilon_i d^\delta_j = d^\delta_{j-1} d^\epsilon_i, \quad 1 \leq i < j \leq n, \ \ \epsilon, \delta = 0, 1,$$

we define a chain complex with chain groups X_n and a boundary map $\partial_n : X_n \to X_{n-1}$ given by:

$$\partial_n = \sum_{i=1}^{n} (-1)^i (d^0_i - d^1_i).$$

Note that here $0, 1, X_n$ corresponds to l, r, X^n, respectively.
Compare[4, 15].

3. Equivalence of Two Homology Theories

Theorem 3.1. *Chain complexes (C_n^{YB}, ∂_n^A) and (C_n^{YB}, ∂_n^G) are chain homotopy equivalent. Consequently, the "algebraic" and "graphic" definitions of Yang–Baxter homology coincide.*

Proof. Consider an n-dimensional cube I^n and denote it as $I_1 \times \cdots \times I_n$. For any coloring, say $L(I_1 \times \cdots \times I_n) = (x_1, \ldots, x_n)$, and an $(n-1)$-face $\mathcal{J} = I_1 \times \cdots \times I_{i-1} \times 1_i \times I_{i+1} \times \cdots \times I_n$, we are going to demonstrate that $L(\mathcal{J}) = d_{i,n}^l(x_1, \ldots, x_n)$. To see this, we need to calculate the coloring of the initial path $(a_1, \ldots, a_{n-1})$ of this $(n-1)$-face. By definition,

$$a_1 = I_1 \times 0_2 \times 0_3 \times \cdots \times 0_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n$$
$$a_2 = 1_1 \times I_2 \times 0_3 \times \cdots \times 0_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n$$
$$\vdots$$
$$a_{i-1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times I_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n$$
$$a_i = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times I_{i+1} \times 0_{i+2} \times \cdots \times 0_n$$
$$a_{i+1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times 1_{i+1} \times I_{i+2} \times \cdots \times 0_n$$
$$\vdots$$
$$a_{n-2} = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times 1_{i+1} \times \cdots \times I_{n-1} \times 0_n$$
$$a_{n-1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times 1_{i+1} \times \cdots \times 1_{n-1} \times I_n.$$

We need another sequence $(b_2, \ldots, b_i)$, where

$$b_2 = 1_1 \times 0_2 \times 0_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n$$
$$b_3 = 1_1 \times 1_2 \times 0_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n$$
$$\vdots$$
$$b_{i-1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n$$
$$b_i = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 1_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n.$$

For example, for $j = i$, the edges of the square are

$$e_{i-1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times I_{i-1} \times 0_i \times 0_{i+1} \times \cdots \times 0_n,$$
$$b_i = e_i = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 1_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n,$$
$$b_{i-1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n,$$
$$a_{i-1} = 1_1 \times 1_2 \times 1_3 \times \cdots \times I_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n.$$

Since

$$L(e_{i-1}) = x_{i-1},\ L(b_i) = x_i,$$

we have

$$L(a_{i-1}) = R_2(L(e_{i-1}), L(b_i)) = R_2(x_{i-1}, x_i),$$
$$L(b_{i-1}) = R_1(L(e_{i-1}), L(b_i)) = R_1(x_{i-1}, x_i).$$

Once we know the color of b_j, we know the colors of b_{j-1}, and a_{j-1} (at each iteration we deal with a square in Fig. 5). Thus, recursively, we know all the colors of $a_j s'$, i.e. we get an $(n-1)$-tuple.

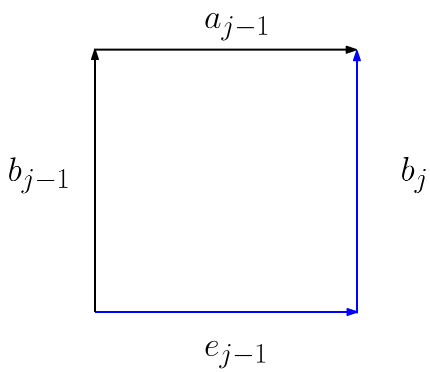

Fig. 5. Square of iteration.

In general, we have

$$L(a_j) = \begin{cases} R_2(L(e_j), L(b_{j+1})) & 1 \leq j \leq i-1, \\ x_{j+1} & i \leq j \leq n-1, \end{cases}$$

$$L(b_j) = \begin{cases} R_1(L(e_j), L(b_{j+1})) & 2 \leq j \leq i-1, \\ x_j & j = i. \end{cases}$$

We can see that this $(n-1)$-tuple is the same as the one given by $d^l_{i,n}$(compare with Fig. 3).

Similarly, if we consider $\mathcal{J} = I_1 \times \cdots \times I_{i-1} \times 0_i \times I_{i+1} \times \cdots \times I_n$, we can show $L(\mathcal{J}) = d^r_{i,n}(x_1, \ldots, x_n)$.

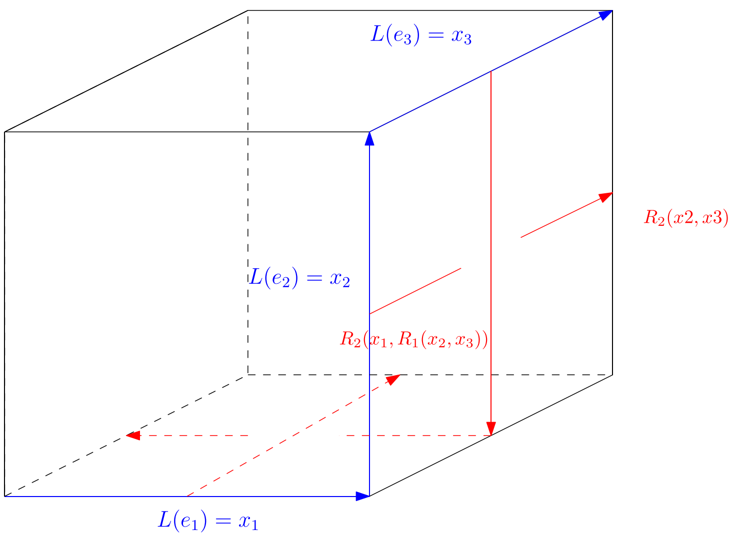

Fig. 6. Coloring of face $I \times I \times 1$.

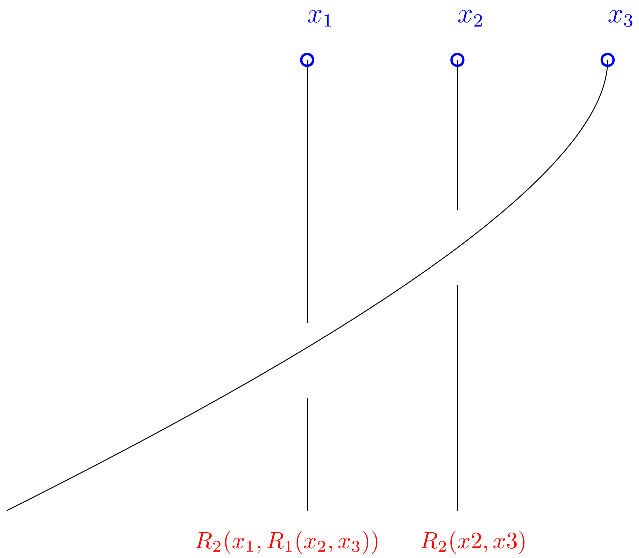

Fig. 7. Face map $d^l_{3,3}$.

As for the sign, we can directly calculate that, for $L(I_1 \times \cdots \times I_{i-1} \times \epsilon_i \times I_{i+1} \times \cdots \times I_n)$, the sign is $(-1)^{n-i}(1 - 2\epsilon_i)$.

Thus, the boundary map of "algebraic" version and the boundary map of "graphic" version only differ by a global sign, therefore, the considered chain complexes are isomorphic and they give isomorphic homology groups. $\square$

Example 3.2. Comparison of the face maps corresponding to the 2-face $I_1 \times I_2 \times 1_3$ of a cube and $d^l_{3,3}$.

From Fig. 6, we can see a "curtain-like" object similar to Fig. 7 "climbing" on the faces of the cube. For higher dimensions, the similar consideration holds. Therefore, this also gives a way to visualize the equivalence.

4. Homology for Unital Yang–Baxter Operators

The "graphic" definition of Yang–Baxter homology was motivated by the homology theory of self-distributive systems [11, 13], for example shelves, racks and quandles. More generally, we can define one-term and two-term homology not only for set-theoretic Yang–Baxter operators but also for pre-Yang–Baxter operators.

Definition 4.1. Let k be a commutative ring and V be a k-module. If a k-linear map, $R : V \otimes V \to V \otimes V$, satisfies the following equation:

$$(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R),$$

then we say R is a pre-Yang–Baxter operator. The equation above is called a Yang–Baxter equation. If, in addition, R is invertible, then we say R is a Yang–Baxter operator. From now on, we assume k is a commutative ring with identity and $V = kX$, the free k-module with a basis set X.

Furthermore, we can extend the definition of pre-Yang–Baxter homology to pre-Yang–Baxter operators with Yang–Baxter wall (see Definition 4.3). We will give the general definitions below and discuss some properties of these homology theories.

4.1. *One-term Yang–Baxter homology*

Let k be a commutative ring with identity, $V = kX$ be a free k-module with basis X and M be a right k-module. We define in Definition 4.3 the one-term pre-Yang–Baxter chain complex $\mathcal{C}^{YB} = (C_n, M, \partial_n)$ from the pre-simplicial module (C_n, M, d_i).

First, we recall the notion of a pre-simplicial module.

Definition 4.2. The pre-simplicial module $\mathcal{M}$ is a collection of modules M_n, $n \geq 0$, together with maps, called face maps or face operators,

$$d_i : M_n \to M_{n-1}, \quad 0 \leq i \leq n,$$

such that

$$d_i d_j = d_{j-1} d_i, \quad 0 \leq i < j \leq n,$$

we define a chain complex with chain modules M_n and a boundary map $\partial_n : M_n \to M_{n-1}$ given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

We are ready to define the pre-Yang–Baxter pre-simplicial module.

Definition 4.3 ([10, 13]). Consider a linear map $R^W : M \otimes V \to M$, such that $R^W \circ (R^W \otimes id_V) = R^W \circ (R^W \otimes id_V) \circ (id_M \otimes R)$ as shown graphically in Fig. 8, we call this the left wall condition.

Let $C_n = M \otimes V^{\otimes n}$ and the face map $d_i = d_{i,n} : C_n \to C_{n-1}$ is defined by

$$\begin{aligned} d_i = {} & (R^W \otimes id^{\otimes n-1}) \circ (id_M \otimes R \otimes id^{\otimes n-2}) \circ (id_M \otimes id \otimes R \otimes id^{\otimes n-3}) \circ \cdots \circ \\ & \times (id_M \otimes id^{\otimes i-2} \otimes R \otimes id^{n-i}). \end{aligned}$$

We can interpret the face maps through Fig. 9.

(C_n, M, d_i) is a pre-simplicial module, and (C_n, M, ∂_n) is the one-term pre-Yang–Baxter chain complex. Its homology is called the one-term pre-Yang–Baxter homology $H_n(R, R^W)$.

The following result generalizes Corollary 8.2(ii) about one-term rack homology in [11] (see Remark 4.7).

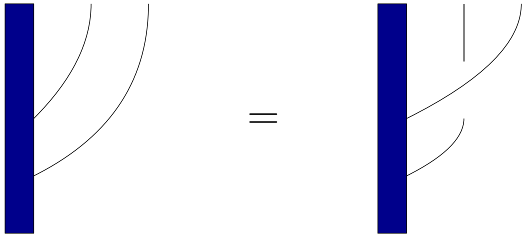

Fig. 8. The left wall condition.

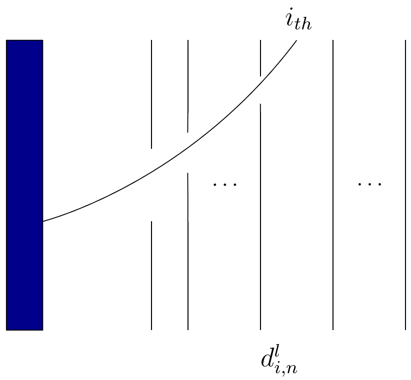

Fig. 9. Face map $d_{i,n}$.

Proposition 4.4. *Let (C_n, M, ∂_n) be the chain complex of the one-term pre-Yang–Baxter homology. For a fixed element $v \in V$, consider the map $f_n : C_n \to C_n$, defined by $f_n(a) = d_{n+1,n+1}(a \otimes v)$, where $a \in C_n$, then this is a chain map. We have*

(1) $(f_n)_*(H_n) = 0$,
(2) *If there is an element $v \in V$ such that f_n is invertible, then the one-term pre-Yang–Baxter homology is trivial.*

Proof. We construct a chain homotopy P_n between $(-1)^{n+1} f_n$ and zero map, in particular, showing that f_n is a chain map. The chain homotopy $P_n : C_n \to C_{n+1}$ is defined by

$$P_n(v_{i_1} \otimes \cdots \otimes v_{i_n}) = (-1)^n v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes v.$$

We check

$$\begin{aligned}\partial_{n+1} P_n + P_{n-1}\partial_n &= \sum_{i=1}^{n+1} (-1)^i d_{i,n+1} P_n + \sum_{j=1}^{n} (-1)^j P_n d_{i,n} \\ &= (-1)^{n+1} d_{n+1,n+1} P_n = (-1)^{n+1} f_n.\end{aligned}$$

(1) Follows because $\{f_n\}$ is chain homotopic to the 0 map, therefore, $(f_n)_*$ is the 0 map on homology. (2) Follows since if f_n is invertible, so is $(-1)^{n+1}f_n$, thus $H_n(C_*)$ is isomorphic to $H_n(C_*)$ through zero map. This shows that $H_n(C_*) = 0$. □

In the case that $M = k$, and V acting on k trivially ($R^W(a, v_i) = a$, where a is in M and v_i is a basis element of V), the left wall condition (in Definition 4.3) is equivalent to that the sum of each column of the R matrix is 1, which we call the column unital condition (e.g. stochastic matrices satisfy the condition).

Corollary 4.5. *Let $M = k$, V act on k trivially, and R be a set-theoretic Yang–Baxter operator. If for any $A_2, A_4 \in X$, there is a unique $A_1 \in X$ such that $R_2(A_1, A_2) = A_4$, then conditions in Proposition* 4.3 *hold. In particular, biracks satisfy this condition (see Definition* 3.1 *condition* 3, *that is right invertibility, in* [2]).

Proof. We need to show that for any n-tuple $(y_1, \ldots, y_n)$ in $C_n = kX^n$, there exists unique n-tuple $(x_1, \ldots, x_n)$ in C_n such that

$$f_n((x_1, \ldots, x_n)) = d_{n+1,n+1}((x_1, \ldots, x_n, v)) = (y_1, \ldots, y_n).$$

Since we know y_n and v, we get the values of x_n and $R_1(x_n, v)$ uniquely. Once we have $R_1(x_n, v)$, together the value of y_{n-1}, we get the value of x_{n-1} and $R_1(x_{n-1}, R_1(x_n, v))$ uniquely. Thus by this iteration, we get the n-tuple $(x_1, \ldots, x_n)$ uniquely and this shows f_n is invertible. □

Example 4.6 (Compare [2]). Let F be a commutative ring with identity. Let $k = F[s^{\pm 1}, t^{\pm 1}]/(1 - s)(1 - t)$, then

$$R(x, y) = (R_1(x, y), R_2(x, y)) = ((1 - s)x + ty, sx + (1 - t)y)$$

is a set-theoretic Yang–Baxter operator satisfying the conditions in Corollary 4.4. This holds because for any given y and $a = R_2(x, y) = sx + (1 - t)y$, we can solve $x = s^{-1}(a - (1 - t)y)$. Thus, the one-term homology of this operator is trivial.

Remark 4.7. Pre-Yang–Baxter coming from racks $(X, *)$ where $R(a, b) = (b, a * b)$ satisfies the conditions in Proposition 4.3. Thus, it has zero one-term homology (see [11]).

4.2. *Two-term Yang–Baxter homology*

Let k be a commutative ring with identity, $V = kX$ be a free k-module with basis X, M be a right k-module and N be a left k-module. We define in Definition 4.9 the two-term pre-Yang–Baxter chain complex $\mathcal{C}^{YB} = (C_n, M, N, \partial_n)$ from the precubical module $(C_n, M, N, d_i^\epsilon)$.

Definition 4.8. The pre-cubical module $\mathcal{M}$ is a collection of modules M_n, $n \geq 0$, together with maps, called face maps or face operators,

$$d_i^\epsilon : M_n \to M_{n-1}, \quad 1 \leq i \leq n, \quad \epsilon = 0, 1$$

such that

$$d_i^\epsilon d_j^\delta = d_{j-1}^\delta d_i^\epsilon, \quad 1 \leq i < j \leq n, \quad \epsilon, \delta = 0, 1,$$

we define a chain complex with chain groups M_n and a boundary map $\partial_n : M_n \to M_{n-1}$ given by

$$\partial_n = \sum_{i=1}^{n} (-1)^i (d_i^0 - d_i^1).$$

We are ready to define the pre-Yang–Baxter pre-cubical module.

Definition 4.9 ([10, 13]). Consider a linear map $R_l^W : M \otimes V \to M$, satisfies the left wall condition in Definition 4.3 (see Fig. 8) and a linear map $R_r^W : V \otimes N \to N$, such that $R_r^W \circ (id_V \otimes R_r^W) = R_r^W \circ (id_V \otimes R_r^W) \circ (R \otimes id_N)$ we call it the right wall condition (see Fig. 10).

Let $C_n = M \otimes V^{\otimes n} \otimes N$ and face maps $d_i^\epsilon : C_n \to C_{n-1}$ are given by

$$\begin{aligned} d_i^l = {} & (R_l^W \otimes id^{\otimes n-1} \otimes id_N) \circ (id_M \otimes R \otimes id^{\otimes n-2} \otimes id_N) \circ \\ & \times (id_M \otimes id \otimes R \otimes id^{\otimes n-3} \otimes id_N) \circ \cdots \circ (id_M \otimes id^{\otimes i-2} \otimes R \otimes id^{\otimes n-i} \otimes id_N) \end{aligned}$$

and

$$\begin{aligned} d_i^r = {} & (id_M \otimes id^{\otimes n-1} \otimes R_r^W) \circ (id_M \otimes id^{\otimes n-2} \otimes R \otimes id_N) \circ (id_M \otimes id^{\otimes n-3} \\ & \otimes R \otimes id \otimes id_N) \circ \cdots \circ (id_M \otimes id^{\otimes i-1} \otimes R \otimes id^{\otimes n-i-1} \otimes id_N). \end{aligned}$$

We can interpret the face maps d_i^l, d_i^r and their difference through Fig. 11.[c]

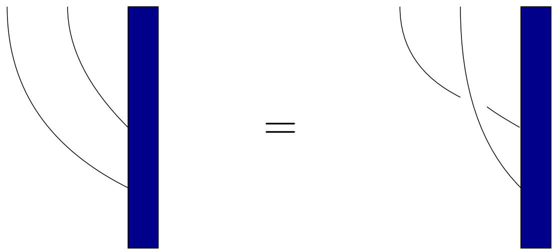

Fig. 10. The right wall condition.

[c]The "curtain" interpretation of face maps was observed by Ivan Dynnikov at Przytycki's talk in Moscow in May of 2002 (see [12]). It was observed few weeks before by Victoria Lebed working on her Ph.D. thesis [9, 10].

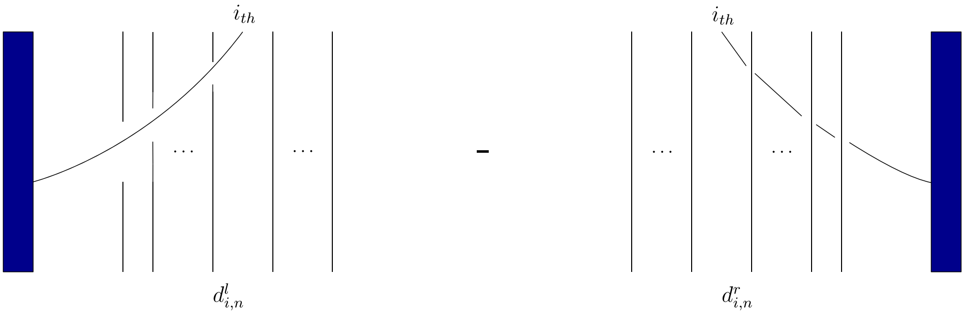

Fig. 11. Face map $d_{i,n}$.

$(C_n, M, N, d_i^\epsilon)$ is a pre-cubical module, and (C_n, M, N, ∂_n) is the two-term pre-Yang–Baxter chain complex. Its homology is called the two-term pre-Yang–Baxter homology $H_n(R, R_l^W, R_r^W)$.

In the case that $M = k = N$, and the action of V on k is the trivial action, the left wall condition and the right wall condition is equivalent to saying that R has the column unital condition.

Example 4.10. We give a family of unital Yang–Baxter operator $R : V \otimes V \to V \otimes V$, where $V = k\{v_1, \ldots, v_m\}$, $k = \mathbb{Q}[y, y^{-1}]$ and m is a positive integer. For any given m, R can be represented by its coefficients,

$$R_{ij}^{kl} = \begin{cases} 1, & \text{if } i = j = k = l, \\ 1, & \text{if } l = i > j = k, \\ y^2, & \text{if } l = i < j = k, \\ 1 - y^2, & \text{if } k = i < j = l, \\ 0, & \text{otherwise.} \end{cases}$$

These family of Yang–Baxter operators are unital, for example, when $m = 2$, it is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - y^2 & 1 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Our computation shows interesting pattern in the two-term Yang–Baxter homology of this family of Yang–Baxter operators.

Conjecture 4.1 ([17]). *When $m = 2$, $H_n = k^2 \bigoplus (k/(1 - y^2))^{a_n} \bigoplus (k/(1 - y^4))^{s_{n-2}}$, where $s_n = \Sigma_{i=1}^{n+1} f_i$ is the partial sum of Fibonacci sequence, where $f_1 = 1 = f_2$ and a_n is given by $2^n = 2 + a_{n-1} + s_{n-3} + a_n + s_{n-2}$ with $a_1 = 0$. We verified the conjecture for $n \leq 11$.*

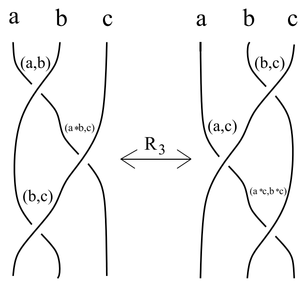

Fig. 12. Relation between third Reidemeister move and 2-(co)cycle condition.

What fascinates us is that this family of Yang–Baxter operators come from the Yang–Baxter operators giving sl_m polynomial invariants of links (substitutions to the Homflypt polynomial) see [5] and [16]. For example, when $m = 2$, the matrix is

$$\begin{bmatrix} -q & 0 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q. \end{bmatrix}.$$

If we divide elements in each column by the sum of those elements and make the substitution $y = (1 + q^{-1} - q)^{-1/2}$, we will get the matrix in Example 4.9. In general, we can get our family in Example 4.9 in a similar way (normalizing columns) and again they are Yang–Baxter operators. More interestingly, this new family of Yang–Baxter operators also provide sl_m polynomial invariants of links [17]. This fact is implicit in [6].

5. Future Work

Cocycle invariant for knotted curves and surfaces were defined in [3]. It was generalized by Carter, Elhamdadi and Saito to set-theoretic Yang–Baxter homology [2]. We plan to investigate the possibility of 2-cocycle invariant in the case of column unital Yang–Baxter operators. In [14], it demonstrates the third Reidemeister move preserve the (co)homology of the (co)cycle constructed form a knot diagram. See Fig. 12 and [14] for details. The goal is to establish connections between Yang–Baxter homology and Khovanov homology and Khovanov–Rozansky homology [7, 8].

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