Biracks and their applications – Part III

Braces and biracks

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Left braces

Definition (W. Rump)

A set B equipped with operations $+$ and $\circ$ is called a *left brace* if

- $(B, +)$ is an abelian group;
- $(B, \circ)$ is a group;
- for all $a, b, c \in B$, we have $a \circ (b + c) = a \circ b + a \circ c - a$.

Example

Let $(R, +, *)$ be a radical ring. Let $a \circ b = a + a * b + b$. Then $(B, +, \circ)$ is a left brace.

Example

Let $(R, +, \cdot)$ be a commutative ring and let $n \in \mathrm{nil}(R)$. Let $a * b = anr$. Then $(R, +, *)$ is a commutative radical ring.

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Two-sided braces

Definition

A left brace is called *two-sided* if $(a+b)\circ c = a\circ c + b\circ c - c$.

Example

Let $\circ$ be commutative. Then the left brace is two-sided.

Proposition (W. Rump)

Let $(B,+,\circ)$ be a two-sided brace. Let $a*b = a\circ b - a - b$. Then $(R,+,*)$ is a radical ring.
Moreover, if B is finite then

$$B = \prod_{p \text{ prime}} B_p,$$

where $B_p = \{b \in B \mid \exists m \in \mathbb{N} : p^m b = 0\}$.

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Semidirect product

Definitions

Let $(A, +_A, \circ_A)$ and $(B, +_B, \circ_B)$ are two left braces. An *action* of B on A is a homomorphism $\phi : (B, \circ_B) \to \mathrm{Aut}(A)$.

The *semidirect product* $A \rtimes_\phi B$ is defined as follows:

$$(a_1, b_1) + (a_2, b_2) = (a_1 +_A a_2, b_1 +_B b_2),$$
$$(a_1, b_1) \circ (b_2, b_2) = (a_1 \circ_A \phi(b_1)(a_2), b_1 \circ_B b_2)$$

Homomorphism λ

Proposition

Let $(B, +, \circ)$ be a left brace. The mapping $\lambda : B \to B^B$ defined by

$$\lambda_a(b) = a \circ b - a$$

is a group homomorphism $(B, \circ) \to \mathrm{Aut}(B, +)$.

Proof.

$$\lambda_a(b + c) = \lambda_a(b) + \lambda_a(c)$$

$$\lambda_a^{-1}(b) = a^{-1} \circ (a + b)$$

$$\lambda_{a \circ b}(c) = \lambda_a \lambda_b(c)$$

□

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Cohomology

Observation

Let $(B,+,\circ)$ be a left brace. The action λ turns $(B,+)$ into a left $(B,\circ)$-module such that the identity is a 1-cocycle.

On the other hand, if we have a group G, a left G-module M and a bijective 1-cocycle $\phi : G \to M$ then, by defining

$$a+b=\phi^{-1}(\phi(a)+\phi(b))$$

we obtain a left brace.

Ideals in left braces

Definition

A subset I of a left brace $(B, +, \circ)$ is called an *ideal* if I is a subgroup of $(B, +)$, I is a normal subgroup of $(B, \circ)$ and $\lambda_a(I) \subseteq I$, for each $a \in B$.

Observation

Ideals of left braces correspond to homomorphism pre-images of 0. On the other hand, every endomorphism is determined by its kernel.

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Socle

Definition

The set

$$\mathrm{Soc}(B) = \{s \in B \mid s + a = s \circ a\}$$

is an ideal of B called the *socle*.

Observation

$\mathrm{Soc}(B) = \mathrm{Ker}\,\lambda$.

Observation

If $\circ$ *is commutative then* $\mathrm{Soc}(B, +, \circ) = \mathrm{Ann}(B, +, *)$.

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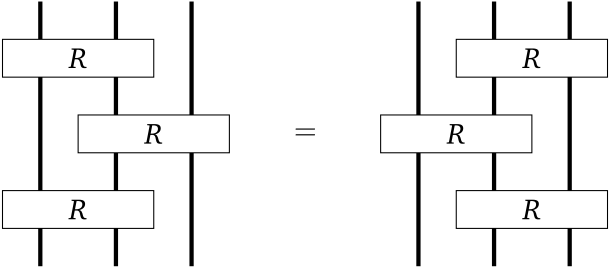
Yang–Baxter equation

Definition

Let V be a vector space. A homomorphism $R : V \otimes V \to V \otimes V$ is called a *solution of Yang–Baxter equation* if it satisfies

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$



Biracks

Definition

An algebra $(B, \triangleleft, \triangleright)$ is called a *birack* if

- $(X, \triangleleft)$ is a left quasigroup,
- $(X, \triangleright)$ is a right quasigroup,
- the mapping $r : (x, y) \mapsto (x \triangleleft y, x \triangleright y)$ is bijective,
- the mapping r satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

Definition

A birack is called *involutive* if $r^2 = \mathrm{id}_{X^2}$.

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Involutive biracks associated to left braces

Proposition

Let $(B, +, \circ)$ *be a left brace. If we define* $\triangleleft$ *and* $\triangleright$ *as*

$$a \triangleleft b = \lambda_a(b)$$
$$a \triangleright b = \lambda^{-1}_{\lambda_a(b)}(a)$$

then $(B, \triangleleft, \triangleright)$ *is an involutive birack.*

Proof.

$\lambda_{\lambda_a(b)}\lambda_{\lambda^{-1}_{\lambda_a(b)}(a))} = \lambda_{\lambda_a(b)\circ\lambda^{-1}_{\lambda_a(b)}(a)} = \lambda_{\lambda_a(b)\circ(\lambda_a(b))^{-1}\circ(\lambda_a(b)+a)} = \lambda_{(a\circ b)-a+a} = \lambda_{a\circ b} = \lambda_a\lambda_b$ □

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Nilpotency of left braces

Definition

Let $(B, +, \circ)$ be a left brace. We define

- $B_0 = B$,
- $B_{i+1} = B_i/\mathrm{Soc}(B_i)$, for $i \geqslant 0$.

We say that B is *nilpotent* of class k if k is the least integer such that $|B_k| = 1$.

Theorem (W. Rump)

A left brace $(B, +, \circ)$ is nilpotent of class k if and only if its associated birack has multipermutation level k

Proof.

$x \sim y \iff \lambda_x = \lambda_y \iff \lambda_{x \circ y^{-1}} = \mathrm{id} \iff x \circ y^{-1} \in \mathrm{Soc}(B)$ □

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Structure group

Definition

Let $(X, \triangleleft, \triangleright)$ be a finite involutive birack. The infinite group with the presentation

$$G_X = \langle X \mid x \circ y = (x \triangleleft y) \circ (x \triangleright y) \rangle$$

is called the *structure group* of the birack X.

Theorem (P. Etingof, T. Schedler, A. Soloviev)

Let $(X, \triangleleft, \triangleright)$ be a finite involutive birack. Then there exists a unique free abelian group operation $+$ on the set G_X such that $(G_X, +, \circ)$ is a left brace and $\lambda_x(y) = x \triangleleft y$, for all $x, y \in X$. In particular, $(X, \triangleleft, \triangleright)$ embeds into the birack associated to G_X.

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Representation of the structure group

Theorem (E. Acri, R. Lutowski, L. Vendramin)

Let X be an involutive birack of size $n \in \mathbb{N}$. Then G_X embeds into $\mathbf{GL}(n+1, \mathbb{Z})$.

Suppose $X = \{x_1, x_2, \ldots, x_n\}$.
For each $a \in G_X$, let A_a be the permutation matrix of the permutation $\lambda_a|_X$ and let $\vec{c}_a$ be such that $a = \sum (c_a)_i x_i$.
We associate to a the matrix $\begin{pmatrix} A_a & \vec{c}_a^T \\ \vec{0} & 1 \end{pmatrix} \in \mathbf{GL}(n+1, \mathbb{Z})$

Proof.
Use $\lambda_a(b) = \sum (A_a \cdot \vec{c}_b^T)_i x_i$ and $a \circ b = a + \lambda_a(b)$. □

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Projection to the multiplication group

Definition

Let (X, r) be an involutive birack. The group

$$\mathrm{Mlt}(X) = \langle L_x \mid x \in X \rangle = \langle L_x, \mathbf{R}_x \mid x \in X \rangle,$$

where $L_x(y) = x \triangleleft y$ and $\mathbf{R}_x(y) = y \triangleright x$, is called the *multiplication* group or the *permutation group* or the *Yang-Baxter group* of $(X, \triangleleft, \triangleright)$.

Let $\pi : G_X \to \mathrm{Mlt}(X)$ send x to L_x. This mapping extends to a homomorphism of groups since $L_x L_y = L_{x \triangleleft y} L_{x \triangleright y}$.

λ_x restricted to X is equal to L_x.

$\lambda_{x_1 \circ x_2 \circ \cdots \circ x_k}$ restricted to X is equal to $L_{x_1} L_{x_2} \cdots L_{x_k}$.

λ_a restricted to X is equal to $\pi(a)$.

$a \in \operatorname{Ker} \lambda \iff \lambda_a$ is the identity $\iff \lambda_a$ is the identity on $X \iff$ $\pi(a)$ is the identity $\iff a \in \operatorname{Ker} \pi$

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Multiplication left brace

Definition

The *multiplication left brace* is the quotient left brace $G_X/\mathrm{Soc}(G_X)$.

Observation

Since $\mathrm{Soc}(G_X) = \mathrm{Ker}\,\lambda$, *the quotient can be viewed as the projection* π *of* G_X *onto* $\mathrm{Mlt}(X)$. *Hence it is usual to consider* $G_X/\mathrm{Soc}(G_X) \cong \mathrm{Mlt}(X)$.

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Since $\lambda_x(y) = L_x(y)$, we have $\pi(\lambda_x(y)) = \pi(L_x(y)) = L_{L_x(y)}$.

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Since $\mathrm{Soc}(G_X) = \mathrm{Ker}\,\lambda$*, the quotient can be viewed as the projection* π *of* G_X *onto* $\mathrm{Mlt}(X)$*. Hence it is usual to consider* $G_X/\mathrm{Soc}(G_X) \cong \mathrm{Mlt}(X)$.

$\pi(x) = L_x$

Since $\lambda_x(y) = L_x(y)$, we have $\pi(\lambda_x(y)) = \pi(L_x(y)) = L_{L_x(y)}$.

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Socle vs. retract

Theorem

Let $(B, +, \circ)$ be a left brace and let $(B, \triangleleft, \triangleright)$ be its associated birack. Let X be a subset of B closed on λ that generates $(B, +)$. Then $\mathrm{Ret}(X) = \{x + \mathrm{Soc}(B) \mid x \in X\} \subseteq B/\mathrm{Soc}(B)$.

Proof.

$$x \sim y \iff \lambda_x = \lambda_y \text{ on } X \iff \lambda_x = \lambda_y \text{ on } B \iff \lambda_x \lambda_y^{-1} = \mathrm{id} \iff x \circ y^{-1} \in \operatorname{Ker} \lambda \iff x \circ y^{-1} \in \mathrm{Soc}(B)$$

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Corollaries of the retract

Corollary

$\sim$ is a congruence for every finite involutive birack.

Proof.

Since X embeds into G_X, $\mathrm{Ret}(X)$ embeds into $\mathrm{Mlt}(X)$. □

Corollary

If X is finite and $\mathrm{Mlt}(X)$ is abelian then X is multipermutation.

Proof.

$\mathrm{Ret}(X)$ embeds into $\mathrm{Mlt}(X)$. Since $(\mathrm{Mlt}(X), \circ)$ is abelian, $(\mathrm{Mlt}(X), +, *)$ is a radical ring. All finite commutative radical rings are nilpotent. Hence $\mathrm{Ret}(X)$ is multipermutation. □

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Indecomposable involutive biracks

Definition

We say that an involutive birack $(X, \triangleleft, \triangleright)$ is *indecomposable* if the group $\mathrm{Mlt}(X)$ is transitive.

Theorem (W. Rump, reformulated by M. Castelli)

Let $(B, +, \circ)$ and let $g \in B$ be such that the orbit of g under the action λ generates the left brace B. If we define

$$a \triangleleft b = \lambda_a(g) \circ b$$

then $(B, \triangleleft, \triangleright)$ is an indecomposable involutive birack with its mutiplication left brace isomorphic to B.

Moreover, every indecomposable involutive birack can be obtained this way.

Invariants of isomorphisms

Theorem (W. Rump)

Let k be a power of prime and let $(B,+,\circ)$ be a left brace of size k with $(B,\circ)$ cyclic. Then $(B,+)$ is cyclic if and only if $k \neq 4$.

Theorem (P. J., A. P., A. Z.-D.)

A complete set of invariants for finite indecomposable involutive biracks with cyclic multiplication groups are

- $k \in \mathbb{N}$,
- $n \in \mathbb{N}$ *such that,*
  - n *divides* k,
  - *every prime* p *divides* n *whenever* p *divides* k,
  - *if 8 divides* k *then 4 divides* n,
- $g \in \{1, \ldots, \gcd(n, k/n)\}$ *coprime to* k.

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Skew left braces

Definition (L. Guarnieri, L. Vendramin)

A set B equipped with operations $+$ and $\circ$ is called a *skew left brace* if

- $(B, +)$ is a group;
- $(B, \circ)$ is a group;
- for all $a, b, c \in B$, we have $a \circ (b + c) = a \circ b - a + a \circ c$.

Example

Let $(B, +)$ be a group. Then $(B, +, +_{op})$ is a skew left brace.

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Biracks associated to skew left braces

Proposition (L. Guarnieri, L. Vendramin)

Let $(B,+,\circ)$ be a skew left brace. The mapping $\lambda : B \to \mathfrak{S}_B$ defined by $\lambda_a(b) = -a + a \circ b$ is a homomorphism $B \to \mathrm{Aut}(B,+)$.

Proposition (D. Bachiller)

Let $(B,+,\circ)$ be a skew left brace. The mapping $\rho : B \to \mathfrak{S}_B$ defined by $\rho_b(a) = (\lambda_a(b))^{-1} \circ a \circ b$ is an anti-homomorphism, that means $\rho_{a\circ b} = \rho_b \rho_a$.

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Let $(B,+,\circ)$ be a left brace. If we define

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Holomorph

Definition

Let G be a group. The *holomorph* of a group is the group $G \rtimes \mathrm{Aut}(G)$ with the operation

$$(g, \alpha) \cdot (h, \beta) = (g\alpha(h), \alpha\beta).$$

A subgroup $H \leqslant \mathrm{Hol}(G)$ is called *regular* if, for each $g \in G$, there exists a unique $\phi_g \in \mathrm{Aut}(G)$ such that $(g, \phi_g) \in H$.

Theorem (L. Guarnieri, L. Vendramin)

There is a 1-1 correspondence between skew left braces and regular subgroups of holomorphs.

Proof.

$b, \lambda_b \leftrightsquigarrow (b, \phi_b)$ □

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Ideals in skew left braces

Definition

A subset I of a skew left brace $(B,+,\circ)$ is called an *ideal* if I is a normal subgroup of $(B,+)$, I is a normal subgroup of $(B,\circ)$ and $\lambda_a(I) \subseteq I$, for each $a \in B$.

Definition

The set

$$\mathrm{Soc}(B) = \{s \in B \mid \forall a \in B\ a+s = s+a = s \circ a\}$$

is an ideal of B called the *socle*.

Proposition (D. Bachiller)

$\mathrm{Soc}(B) = \mathrm{Ker}\,\lambda \cap \mathrm{Ker}\,\rho$

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Nilpotency of left braces

Definition

Let $(B, +, \circ)$ be a skew left brace. We define

- $B_0 = B$,
- $B_{i+1} = B_i / \mathrm{Soc}(B_i)$, for $i \geqslant 0$.

We say that B is *socle-nilpotent* of class k if k is the least integer such that $|B_k| = 1$.

Theorem (D. Bachiller)

A skew left brace $(B, +, \circ)$ is socle-nilpotent of class k if and only if its associated birack has multipermutation level k.

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Central nilpotency

Definition (I. Colazzo, F. Catino, P. Stefanelli)

The *annihilator* or the *center* of a skew left brace is the ideal

$$\{c \in B \mid \forall a \in B \quad c + a = a + c = c \circ a = a \circ c\}$$

or simply $Z(B,+) \cap Z(B,\circ) \cap \operatorname{Ker}\lambda$.

Definitions (M. Bonatto, P. J.)

Upper central series:

$$Z_0(B) = 0, \quad Z_n = \{c \in B \mid \forall a \in B \quad a * c, c * a, [a,c]_+ \in Z_{n-1}(B)\},$$

Lower central series:

$$\Gamma_0(I) = I, \quad \Gamma_n(I) = \langle \Gamma_{n-1}(I) * B, B * \Gamma_{n-1}(I), [\Gamma_{n-1}(I), B]_+ \rangle_+,$$

where $x * y = -x + (x \circ y) - y$.

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Commutator in skew braces

Theorem (D. Bourn, A. Facchini, M. Pompili)

The commutator of two ideals I and J in a skew brace $(B, +, \circ)$ is the smallest ideal generated by $[I, J]_+$, $[I, J]_\circ$ and $I * J$.

Corollary

$$Z_n(B)/Z_{n-1}(B) = Z(B/Z_{n-1}(B))$$

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Opposite skew left braces

Proposition

Let $(B, \triangleleft, \triangleright)$ be a birack and let $r(x,y) = (x \triangleleft y, x \triangleright y)$. Then $r^{-1} : (x,y) \mapsto (x \hat{\triangleleft} y, x \hat{\triangleright} y)$ is a birack.

Definition (A. Koch, P. J. Truman)

Let $(B, +, \circ)$ be a skew left brace. Then $(B, +_{op}, \circ)$ is a skew left brace called the *opposite* skew left brace.

Theorem (A. Koch, P. J. Truman)

The birack associated to $(B, +_{op}, \circ)$ is inverse to the birack associated to $(B, +, \circ)$.

Corollary

$$\hat{\lambda}_a(b) = (a \circ b) - a, \qquad \hat{\rho}_b(a) = (\hat{\lambda}_a(b))^{-1} \circ a \circ b.$$

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Structure group

Definition

Let $(X, \triangleleft, \triangleright)$ be a finite birack. The infinite group with the presentation

$$G_X = \langle X \mid x \circ y = (x \triangleleft y) \circ (x \triangleright y) \rangle$$

is called the *structure group* of the birack X.

Theorem

Let $(X, \triangleleft, \triangleright)$ be a birack. Let

$$A_X = \langle X \mid x + y = y + (y \triangleleft (y \setminus_{\triangleleft} x)) \rangle .$$

Then there exists a bijection $\phi : A_X \to G_X$ such that $\phi(x) = x$ and

$$\phi(a) \circ \phi(b + c) = \phi(\phi^{-1}(\phi(a) \circ \phi(b)) - a + \phi^{-1}(\phi(a) \circ \phi(c))).$$

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Let $(X, \triangleleft, \triangleright)$ be a birack. Let

$$A_X = \langle X \mid x + y = y + (y \triangleleft (y \backslash_{\triangleleft} x)) \rangle .$$

Then there exists a bijection $\phi : A_X \to G_X$ such that $\phi(x) = x$ and

$$\phi(a) \circ \phi(b + c) = \phi(\phi^{-1}(\phi(a) \circ \phi(b)) - a + \phi^{-1}(\phi(a) \circ \phi(c))).$$

Injective biracks

Theorem

Let $(X, \triangleleft, \triangleright)$ *be a birack and let* G_X *be its structure group. If we define*

$$\iota : (X, \triangleleft, \triangleright) \to (G_X, \triangleleft, \triangleright), \qquad x \mapsto x,$$

then ι *is a homomorphism of biracks.*

Corollary

Suppose $\iota(x) = \iota(y)$ *then* $x \sim y$.

Definition

We say that a birack is *injective* if ι is injective.

Observation

Every involutive birack is injective.

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then ι is a homomorphism of biracks.

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Bi-skew left braces

Definition (L. Childs)

A skew left brace $(B, +, \circ)$ is called a bi-skew left brace if $(B, \circ, +)$ is a skew left brace as well.

Theorem (A. Caranti)

A skew left brace $(B, +, \circ)$ is a bi-skew left brace if and only if λ is an anti-homomorphism of $(B, +)$, i.e. $\lambda_{a+b} = \lambda_b \lambda_a$.

Theorem (L. Stefanello, S. Trappeniers)

Let $(B, +, \circ)$ be a skew left brace. Then B is a bi-skew left brace if and only if

$$\lambda_{\hat{\lambda}_a(b)} = \lambda_b,$$

for each $a, b \in B$.

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Distributive biracks

Theorem (P. J., A. Pilitowska)

Let (X,σ,τ) be a birack. TFAE:

- $L_{\hat{L}_x(y)} = L_y$,
- $L_{\mathbf{R}_x(y)} = L_y$,
- $L_xL_y = L_{L_x(y)}L_x$,
- $\hat{L}_x = L_x^{-1}$,
- $L_x \in \mathrm{Aut}(X)$,

for all $x,y \in X$.

Corollary

Let $(B,+,\circ)$ be a skew left brace. TFAE:

- *B is a bi-skew left brace,*
- $\lambda_{a+b} = \lambda_b\lambda_a$,
- $\lambda_{\hat{\lambda}_a(b)} = \lambda_b$,
- $\lambda_{\rho_a(b)} = \lambda_b$,
- $\lambda_a\lambda_b = \lambda_{\lambda_a(b)}\lambda_a$,
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- $\lambda_a \in \mathrm{Aut}(B)$,

for all $a, b \in B$.

Equations of 2-reductivity and skew braces

Proposition (P. J., A. Pilitowska)

Let $(B, +, \circ)$ be a skew left brace. Then

- $\lambda_{\lambda_a(b)} = \lambda_b$ *if and only if* λ *is a homomorphism* $(B, +) \to \mathrm{Aut}(B, \circ)$, *that means* $\lambda_{a+b} = \lambda_a \lambda_b$;
- $\lambda_{\rho_a(b)} = \lambda_b$ *if and only if* λ *is an anti-homomorphism* $(B, +) \to \mathrm{Aut}(B, \circ)$, *that means* $\lambda_{a+b} = \lambda_b \lambda_a$;
- $\rho_{\rho_a(b)} = \rho_b$ *if and only if* ρ *is a homomorphism* $(B, +) \to \mathfrak{S}_X$, *that means* $\rho_{a+b} = \rho_a \rho_b$;
- $\rho_{\lambda_a(b)} = \rho_b$ *if and only if* ρ *is an anti-homomorphism* $(B, +) \to \mathrm{Aut}(B, \circ)$, *that means* $\rho_{a+b} = \rho_b \rho_a$.

Skew left braces and 2-reductivity

Theorem (P. J., A. Pilitowska)

Let $(B, +, \circ)$ *be a skew left brace. TFAE*

- *the birack* $(B, \triangleleft, \triangleright)$ *is* 2*-reductive,*
- $\lambda_{a+b} = \lambda_{b+a} = \lambda_a\lambda_b$ *and* $\rho_{a+b} = \rho_{b+a} = \rho_a\rho_b$*,*
- $(B, \triangleleft, \triangleright)$ *has multipermutation level at most* 2*,*
- $(B, +, \circ)$ *is socle-nilpotent of degree at most* 2*,*
- $(B, +_{op}, \circ)$ *is socle-nilpotent of degree at most* 2*.*

Proposition (P. J., A. Pilitowska)

Let $(X, \triangleleft, \triangleright)$ *be* 2*-reductive. Then* G_X *is socle-nilpotent of degree at most* 2*.*

Skew left braces and 2-reductivity

Theorem (P. J., A. Pilitowska)

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Proposition (P. J., A. Pilitowska)

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