Associative triples and quadratic orthomorphisms

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Inc. joint work with Aleš Drápal and Jack Allsop

Diagonally Cyclic Latin Squares

A Latin square is *diagonally cyclic* if the symbols occur in cyclic order along each broken diagonal parallel to the main diagonal.

$$\begin{bmatrix} 0 & 2 & 5 & 1 & 6 & 4 & 3 \\ 4 & 1 & 3 & 6 & 2 & 0 & 5 \\ 6 & 5 & 2 & 4 & 0 & 3 & 1 \\ 2 & 0 & 6 & 3 & 5 & 1 & 4 \\ 5 & 3 & 1 & 0 & 4 & 6 & 2 \\ 3 & 6 & 4 & 2 & 1 & 5 & 0 \\ 1 & 4 & 0 & 5 & 3 & 2 & 6 \end{bmatrix}$$

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But which first rows work?

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So we want $\theta(a) - a \neq \theta(b) - b$ for all a, b.

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There is a DCLS with first row $[\theta(0), \theta(1), \ldots, \theta(n{-}1)]$ iff θ is an orthomorphism of $\mathbb{Z}_n$.

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Eg. in $\mathbb{Z}_{13}$:

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So θ is a *quadratic orthomorphism*.

Cyclotomic maps

Let γ be a primitive element of the finite field $\mathbb{F}$. For $0 \leqslant j \leqslant k-1$ define the *cyclotomy class* $C_j = \{\gamma^{ki+j} : 0 \leqslant i \leqslant m-1\}$ to be a coset of the unique subgroup C_0 of index k in $\mathbb{F}^*$. A *cyclotomic map* $\phi = \phi_\gamma[a_0, \ldots, a_{k-1}]$ of index k can then be defined by

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ a_i x & \text{if } x \in C_i, \end{cases}$$

where $a_0, \ldots, a_{k-1} \in \mathbb{F}$.

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Such ϕ will be a permutation iff $C_j \mapsto a_j C_j$ permutes the cyclotomy classes.

Quadratic quasigroups $Q_{a,b}$

In general, a quadratic orthomorphism has the form

$$\theta(x) = \begin{cases} ax & \text{if } x \text{ is a square,} \\ bx & \text{if } x \text{ is a nonsquare,} \end{cases}$$

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[We need odd characteristic, and both ab and $(a-1)(b-1)$ to be nonzero squares]

Maximally non-associative quasigroups

Recall: A quasigroup is *maximally non-associative* if $(xy)z = x(yz)$ only when $x = y = z$. Such quasigroups apparently have some application in cryptography for designing second pre-image resistant hash functions.

$Q_{2,5}$

*	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	2	10	6	8	12	4	9	1	5	7	3	11
1	12	1	3	11	7	9	0	5	10	2	6	8	4
2	5	0	2	4	12	8	10	1	6	11	3	7	9
3	10	6	1	3	5	0	9	11	2	7	12	4	8
4	9	11	7	2	4	6	1	10	12	3	8	0	5
5	6	10	12	8	3	5	7	2	11	0	4	9	1
6	2	7	11	0	9	4	6	8	3	12	1	5	10
7	11	3	8	12	1	10	5	7	9	4	0	2	6
8	7	12	4	9	0	2	11	6	8	10	5	1	3
9	4	8	0	5	10	1	3	12	7	9	11	6	2
10	3	5	9	1	6	11	2	4	0	8	10	12	7
11	8	4	6	10	2	7	12	3	5	1	9	11	0
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3	10	6	1	3	5	0	9	11	2	7	12	4	8
4	9	11	7	2	4	6	1	10	12	3	8	0	5
5	6	10	12	8	3	5	7	2	11	0	4	9	1
6	2	7	11	0	9	4	6	8	3	12	1	5	10
7	11	3	8	12	1	10	5	7	9	4	0	2	6
8	7	12	4	9	0	2	11	6	8	10	5	1	3
9	4	8	0	5	10	1	3	12	7	9	11	6	2
10	3	5	9	1	6	11	2	4	0	8	10	12	7
11	8	4	6	10	2	7	12	3	5	1	9	11	0
12	1	9	5	7	11	3	8	0	4	6	2	10	12

is the smallest MNQ built from a quadratic orthomorphism.

Characterisation of quadratic MNQs

$Q_{a,b}$ is an MNQ iff

(1) $a^2 \neq b$ or $a \neq 2b - b^2$,

(2) at least one of -1, $a - 1$ or a is nonsquare,

(3) at least one of b, $(1-a)(a^2-b)$ or $\sigma(a-1)$ is square,

(4) at least one of $a\nu$, $1 - b$ or $a\tau$ is square,

(5) -1 is nonsquare or $\sigma a(b-1)$ is square or $\tau a(b-1)$ is square,

(6) -1 is square or $b - 1$ is nonsquare or $(ab - a + b)b$ is nonsquare,

(7) $(b - a^2)\mu$ is square or $b\mu(ab - 2a + 1)$ is nonsquare or $(a-1)(ab-a+b)\mu$ is square,

(8) -1 is square or $a - 1$ is square or b is nonsquare,

(9) at least one of -1, a or $(ab - 2a + 1)(b - 1)$ is square, and

(10) conditions $(1) - (9)$ all apply when a and b are interchanged.

Here $\mu = b^2 - 2b + a$, $\nu = a^2 - 2a + b$, $\sigma = a^2 b - a^2 - ab + b$, and $\tau = a^2 b - ab - a + b$.

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Using Weil bounds we were able to show that these conditions are satisfied in all large fields (of odd characteristic).

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We also found MNQs from orthomorphisms of these groups:

$\mathbb{Z}_{21}, \mathbb{Z}_{33}, \mathbb{Z}_{35}, \mathbb{Z}_{55},$

$\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_{14}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4.$

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$\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_{14}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$.

Theorem: MNQ exist for $n \geqslant 9$, with the possible exception of
$n \in \{11, 12, 15, 40, 42, 44, 56, 66, 77, 88, 90, 110\}$
and orders of the form $n = 2p_1$ or $n = 2p_1p_2$
for odd primes p_1, p_2 with $p_1 \leqslant p_2 < 2p_1$.

How many quadratic orthomorphisms give MNQs?

Theorem: For odd prime powers q the asymptotic proportion of quadratic orthomorphisms which produce MNQs is

$$\begin{cases} \frac{953}{2^{15}} \approx 0.02908 & \text{for } q \equiv 1 \bmod 4, \\ \frac{825}{2^{16}} \approx 0.01259 & \text{for } q \equiv 3 \bmod 4. \end{cases}$$

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Hence it is viable to find large MNQs "randomly".

We currently do not have a corresponding density result for the near-field construction that Aleš talked about yesterday. Drápal and Lisonek conjecture an asymptotic density of ≈ 0.29.

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Drápal and Hora [2020] built a loop of order 20 by prolonging a cubic quasigroup of order 19. Their loop had $1160 = 3n^2 - 2n$ associative triples, which is the fewest possible for involutory loops.
For all primes $p \geqslant 13$ except $p = 19$ they had been able to find an involutory loop of order $n = p + 1$ with only $3n^2 - 2n$ associative triples by prolonging a quadratic quasigroup.

Automorphisms of quadratic quasigroups

Theorem: Let $Q_{a,b}$ and $Q_{c,d}$ be quadratic quasigroups over $\mathbb{F}$. Then $Q_{a,b}$ is isomorphic to $Q_{c,d}$ if and only if there exists $\alpha \in \mathrm{aut}(\mathbb{F})$ such that $\{a, b\} = \{\alpha(c), \alpha(d)\}$.

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Open question: The corresponding result with "isotopic" in place of "isomorphic".

Theorem: Let $Q = Q_{a,b}$ be a quadratic quasigroup over $\mathbb{F}$ with $a \neq b$. Denote by $\mathbb{K}$ the least subfield of $\mathbb{F}$ that contains $\{a, b\}$. The automorphism group of Q consists of all affine semilinear mappings $x \mapsto \lambda\alpha(x) + \mu$, where λ is a square in $\mathbb{F}^*$, $\mu \in \mathbb{F}$ and $\alpha \in \mathrm{Gal}(\mathbb{F} \mid \mathbb{K})$, except:

(i) If $b = a^\gamma$ and $\gamma^2 = |\mathbb{K}|$, then we also have automorphisms $x \mapsto \lambda\alpha(x^\gamma) + \mu$, where λ is a nonsquare.

(ii) If $|\mathbb{F}| = 7$ and $\{a, b\} = \{3, 5\}$, then $\mathrm{aut}(Q) \cong \mathrm{PSL}_2(7)$.

Quadratic quasigroups in certain varieties

Theorem: Let $Q = Q_{a,b}$ be a quadratic quasigroup upon $\mathbb{F}$. Then

(i) Q is medial (i.e. fulfils the law $xy \cdot uv = xu \cdot yv$) if and only if $a = b$;

(ii) Q is left distributive (i.e. fulfils the law $x \cdot yz = xy \cdot xz$) if and only if $a = b$;

(iii) Q is right distributive (i.e. fulfils the law $xy \cdot z = xz \cdot yz$) if and only if $a = b$;

(iv) Q is commutative if and only if $a + b = 1$ and either $|\mathbb{F}| \equiv 3 \bmod 4$ or $a = b$.

(v) Q is flexible (i.e. fulfils the law $x \cdot yx = xy \cdot x$) if and only if $a = b$ or $\chi(a) = \chi(1 - a) = 1$ or both $a + b = 1$ and $|\mathbb{F}| \equiv 3 \bmod 4$;

(vi) Q is semisymmetric (i.e. fulfils the law $xy \cdot x = y$) if and only if $a^2 - a + 1 = 0$ and either $a = b$ or $a + b = 1$.

(vii) Q is a Steiner quasigroup (i.e. idempotent, commutative and semisymmetric) if and only if either $\mathbb{F}$ has characteristic 3 and $a = b = -1$, or $\mathbb{F}$ has characteristic > 3, $a + b = ab = 1$, and $\chi(a) = \chi(-1) = -1$. In the latter case, $a \neq b$.

(viii) Q is isotopic to a group if and only if $a = b$.

Snow Design

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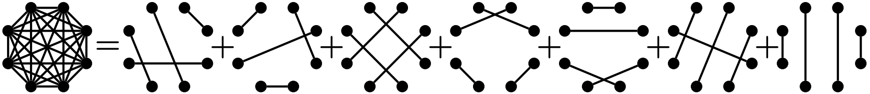

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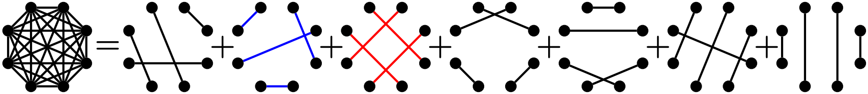

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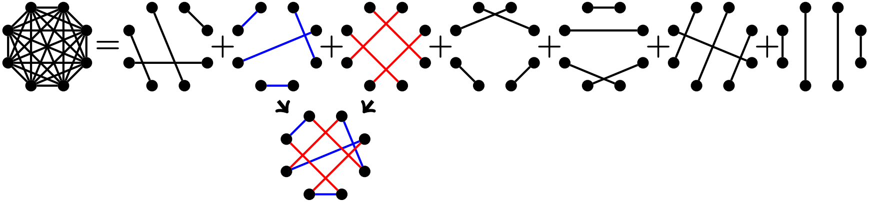

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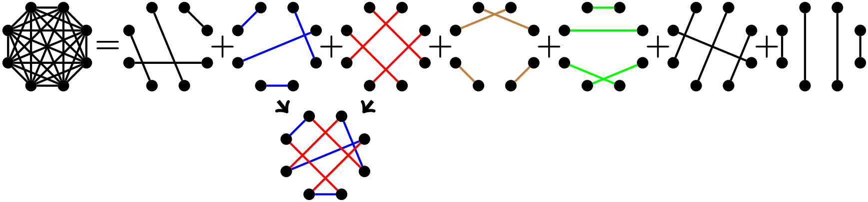

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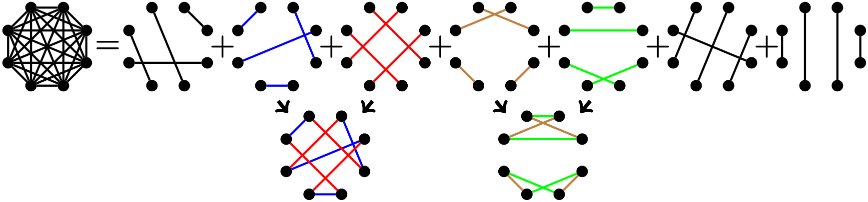

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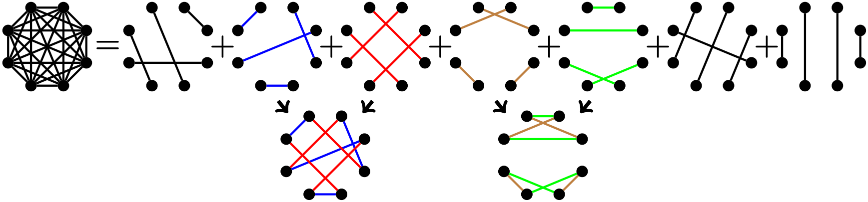

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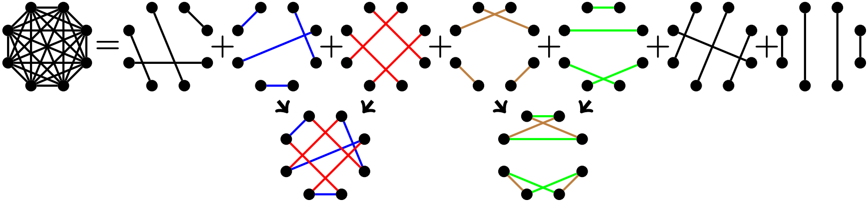

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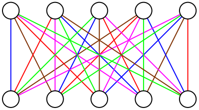
Today I will talk about P1Fs of the complete bipartite graph $K_{n,n}$ (n odd or $n = 2$).

A 1-factorisation of $K_{5,5}$



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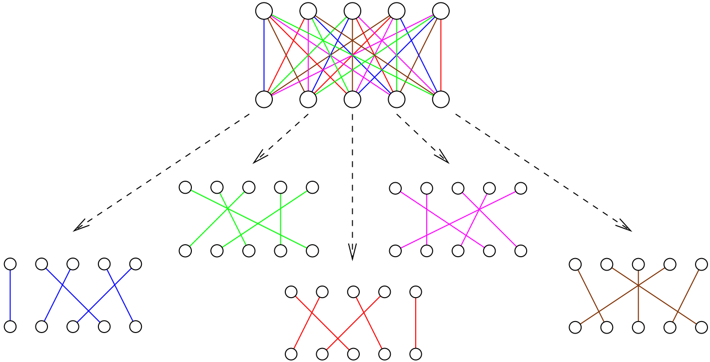

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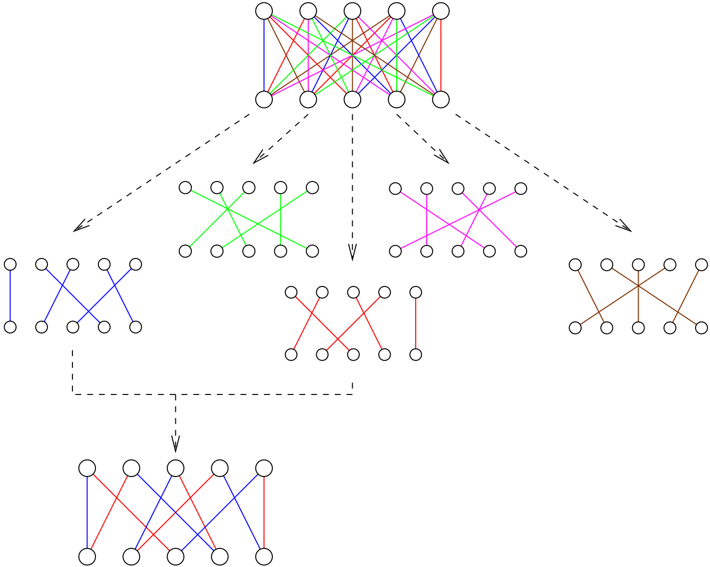
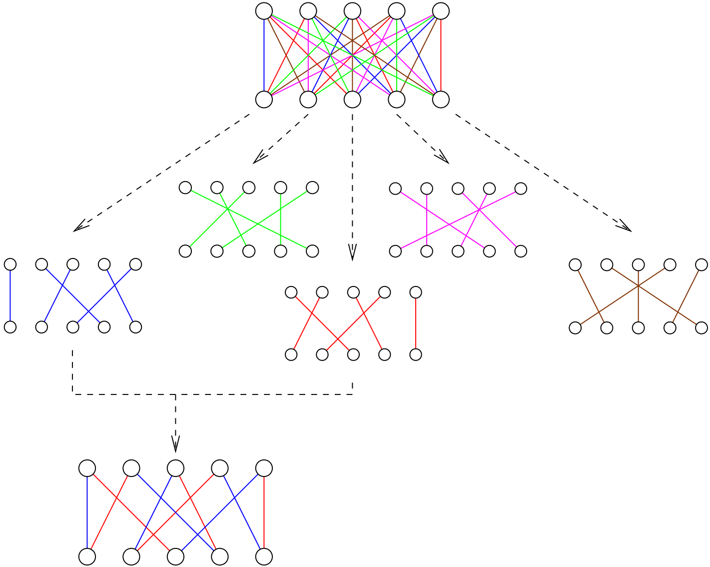


A perfect 1-factorisation of $K_{5,5}$



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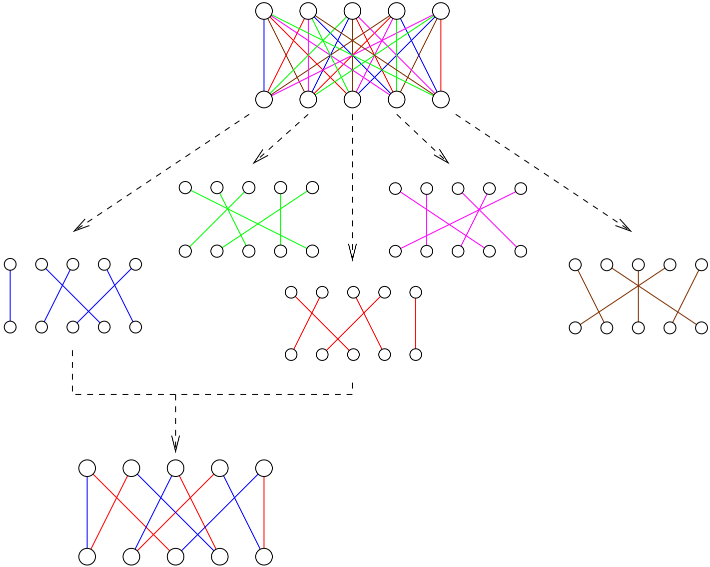

Row Cycles

Two rows of a LS define a permutation, which decomposes into cycles.

	0	2	5	1	6	4	3
	4	1	3	6	2	0	5
	6	5	2	4	0	3	1
$\rightarrow$	2	0	6	3	5	1	4
$\rightarrow$	5	3	1	0	4	6	2
	3	6	4	2	1	5	0
	1	4	0	5	3	2	6

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The rows marked with $\rightarrow$ form the permutation (254)(03)(61). Each of these 3 cycles gives us a *row cycle* (one of which is shown in green).

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The rows marked with $\rightarrow$ form the permutation (254)(03)(61). Each of these 3 cycles gives us a *row cycle* (one of which is shown in green).

Each row cycle corresponds to a cycle of the permutation $L_y \circ L_x^{-1}$ where

$$L_x : Q \rightarrow Q, \qquad L_x(z) = x \cdot z.$$

Similarly, there are *column cycles*, corresponding to cycles of $R_y \circ R_x^{-1}$, where

$$R_x : Q \rightarrow Q, \qquad R_x(z) = z \cdot x.$$

Hamiltonian LS

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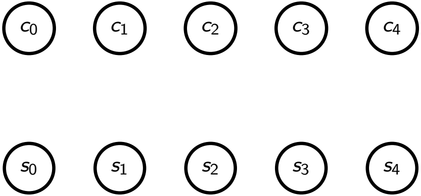
We call L *atomic* if $\nu(L) = 6$.

We have 5 families of atomic Latin squares but all are for prime orders only. There are some sporadic orders up to 39601 known, but they are all for prime power orders.

1-factorisations from Latin squares

1	2	4	0	3
3	1	0	2	4
4	3	2	1	0
0	4	1	3	2
2	0	3	4	1



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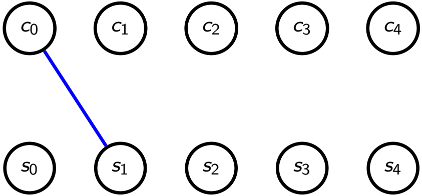

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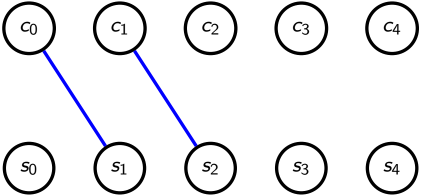

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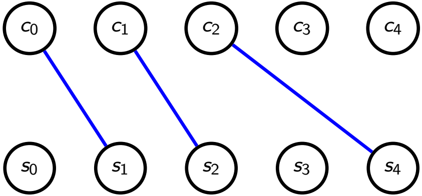

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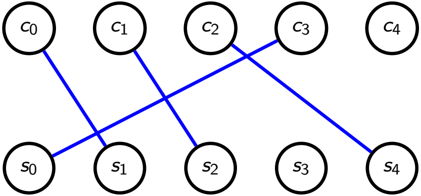

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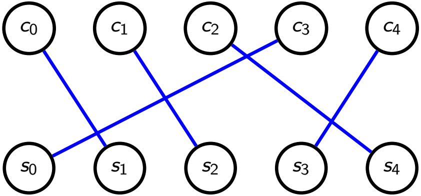

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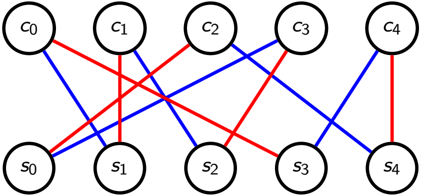

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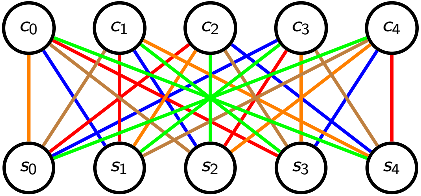

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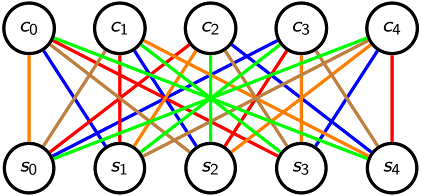

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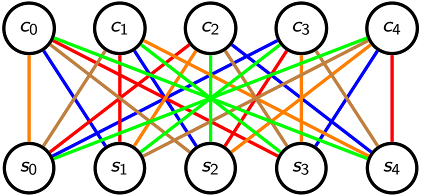

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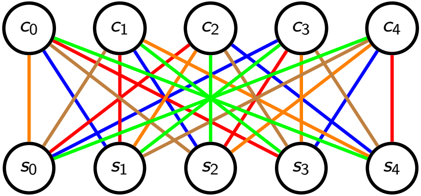

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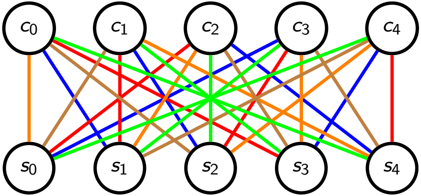

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Only proved for some sporadic orders and $K_{p,p}$, $K_{2p-1,2p-1}$ and K_{p^2,p^2} where p is an odd prime.

Enumeration

n	All 1F of $K_{n,n}$	P1F
2	1	1
3	1	1
4	2	-
5	2	1
6	17	-
7	324	2
8	842227	-
9	57810418543	37
10	104452188344901572	-
11	6108088657705958932053657	

Counted by

6	Clausen 1842??, Tarry 1900	
9		W. 1999
10	McKay/Meynert/Myrvold 2007	
11	Hulpke/Kaski/Östergård 2011	

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Theorem: $\mathcal{L}_p$ is row-Hamiltonian for all $p \equiv 1, 3 \bmod 8$.
It has no Hamiltonian column cycles unless $p \in \{3, 19\}$.

Example $p = 11$

$$\begin{array}{c|cccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
4 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
6 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
7 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
8 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
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5	0	0	0	0	0	1	1	1	1	1
6	0	1	1	1	1	0	1	1	0	1
7	0	1	1	1	1	1	0	1	0	1
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Phase 1

	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	1
2	0	0	1	1	0	1	1	1	1	1
3	0	1	0	0	0	1	1	1	0	1
4	0	1	0	0	0	1	1	1	1	1
5	0	0	0	0	0	1	1	1	1	1
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Phase 1

$$\begin{array}{c c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 9 & 10 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 9 \\ 10 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Phase 2

	1	2	3	4	5	6	9	10
1	0	0	0	0	0	0	0	1
2	0	0	1	1	0	1	1	1
3	0	1	0	0	0	1	0	1
4	0	1	0	0	0	1	1	1
5	0	0	0	0	0	1	1	1
6	0	1	1	1	1	0	0	1
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Phase 2

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 9 \\ 10 \end{array}
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\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
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Phase 3

	1	5	9	10
1	0	0	0	1
5	0	0	1	1
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Phase 3

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The end game

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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Such an x exists for all large p by Weil's theorem, and all small p can be checked by computer.

Not atomic

$\mathcal{L}_p$ has a autoparastrophy mapping its rows to symbols, so $\nu(\mathcal{L}_p) \geqslant 4$.

If there exists $x \in \mathbb{Z}_p$ such that

$$x, x+1, x+\frac{1}{2} \in \Box \quad \text{and} \quad -\frac{x}{2}-\frac{3}{4}, \frac{1}{4}-\frac{x}{2} \notin \Box$$

then $\mathcal{L}_p$ has a column cycle of length 3.

Such an x exists for all large p by Weil's theorem, and all small p can be checked by computer.

$\mathcal{L}_p$ is atomic for $p \in \{3, 19\}$, but otherwise $\nu(\mathcal{L}_p) = 4$.

Varieties

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Question: [Falconer'70] Does there exist a non-trivial, anti-associative, isotopically-closed loop variety?

Solving Falconer's problem

Recall

$$L_x : Q \to Q, \qquad L_x(z) = x \cdot z,$$
$$R_x : Q \to Q, \qquad R_x(z) = z \cdot x.$$

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Consider the variety defined by

$$(L_y \circ L_x^{-1})^p(z) = z, \tag{1}$$

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(1) $\leftrightarrow$ row-Hamiltonian $\leftrightarrow$ every element of every loop isotope has left order dividing p.

(2) $\leftrightarrow$ no column cycle a p-cycle $\leftrightarrow$ every element of every loop isotope has right order coprime to p.

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Theorem: There are infinitely many Falconer varieties.

Loose ends

Question: When is $Q_{a,b}$ isotopic to $Q_{c,d}$?

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Question: For which (a, b, n) do there exist quasigroups of order n in which every row cycle has length a and every column cycle has length b?

That's all. Any questions?

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