Nilpotent structures in topological dynamics, ergodic theory and combinatorics

Schedule and Abstracts

Bedlewo
Poland
June 4-9, 2023

Schedule

Monday, June 5, 2023

09:00-09:45 – **Minicourse on nilspaces (part 1)**, *Introduction to nilspaces and connections to Gowers norms*

10:00-10:45 – **Minicourse on nilspaces (part 2)**, *Nilspaces in an infinitary setting, limit theories, ultraproducts*

10:45-11:15 – *Coffee Break*

11:15-12:00 – **Minicourse on nilspaces (part 3)**, *Nilspaces in topological dynamics*

12:15-13:00 – **Asgar Jamneshan**, *A counterexample to a conjecture of Bergelson, Tao, and Ziegler*

13:15-14:15 – *Lunch*

14:30-15:00 – *Coffee Break*

15:00-15:45 – **Diego González-Sánchez**, *Free nilspaces, double-coset nilspaces, and Gowers norms*

16:00-16:45 – **Daniel Altman**, *A non-flag arithmetic regularity lemma and counting lemma*

17:00-17:45 – **Luka Milićevic**, *Approximate quadratic varieties*

19:00 – *Dinner*

Tuesday, June 6, 2023

09:00-09:45 – **XiangDong Ye**, *A generalization of Furstenberg-Glasner's result to polynomials*

10:00-10:45 – **Joel Moreira**, *Infinite arithmetic configurations in sets with positive density*

10:45-11:15 – *Coffee Break*

11:15-12:00 – **Bryna Kra**, *Infinite sumsets and cubic measures*

12:15-13:00 – **Donald Robertson**, *Disintegrating cubic measures for sumsets*

13:15-14:15 – *Lunch*

14:30-15:00 – *Coffee Break*

15:00-15:45 – **Ethan Ackelsberg**, *Khintchine-type multiple recurrence in abelian groups*

16:00-16:45 – **Or Shalom**, *Inverse theorems in ergodic theory and additive combinatorics*

17:00-17:45 – **Konstantinos Tsinas**, *Multiple ergodic theorems along sequences of polynomial growth*

19:00 – *Dinner*

Wednesday, June 7, 2023

09:00-09:45 – **Wenbo Sun**, *Modern tools in the joint ergodicity problem*

10:00-10:45 – **Sebastian Donoso**, *Joint ergodicity beyond polynomials*

10:45-11:00 – *Coffee Break*

11:00-11:45 – **Andreas Koutsogiannis**, *Convergence of multiple ergodic averages for totally ergodic systems*

12:00-12:45 – **Borys Kuca**, *Multiple ergodic averages along polynomials for systems of commuting transformations*

12:50-13:50 – *Lunch*

14:30-18:00 – *Excursion*

19:00 – *Barbeque*

Thursday, June 8, 2023

09:00-09:45 – **Freddie Manners**, *Cubespaces and homotopy theory*

10:00-10:45 – **Florian Richter**, *Uniformity norms and Hindman's conjecture*

10:45-11:15 – *Coffee Break*

11:15-12:00 – **Sarah Peluse**, *Bounds for sets without L-shapes*

12:15-13:00 – **Joni Teräväinen**, *On Elliott's conjecture and applications*

13:15-14:15 – *Lunch*

14:30-15:00 – *Coffee Break*

15:00-15:45 – **Jakub Konieczny**, *Automatic sequences from the point of view of higher order Fourier analysis*

16:00-16:45 – **John Griesmer**, *Separating recurrence properties*

17:00-17:45 – **Matthew Tointon**, *Approximation of vertex-transitive graphs by nilpotent groups, and applications to probability*

19:00 – *Dinner*

Friday, June 9, 2023

09:00-09:45 – **Tomasz Downarowicz**, *Multiorders and asymptotic pairs in actions of amenable groups*

09:50-10:35 – **Alejandro Maass**, *On recurrence and nilsystems*

10:40-11:25 – **Máté Wierdl**, *Distribution of translation orbits along good averaging sequences*

11:30-12:15 – **Vitaly Bergelson**, *Ergodic Ramsey theory in the nilpotent setup*

12:20-13:20 – *Lunch*

19:00 – *Dinner*

Abstracts

Ethan Ackelsberg, IAS, USA.

Khintchine-type multiple recurrence in abelian groups

Seeking a common extension of Khintchine's recurrence theorem and Furstenberg's multiple recurrence theorem, Bergelson-Host-Kra (2005) proved that for any ergodic measure-preserving system (X, μ, T), any set $A \subseteq X$ with $\mu(A) > 0$, any $\varepsilon > 0$, and $k \leq 3$, there exists (syndetically many) $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > \mu(A)^{k+1} - \varepsilon$ (and this property may fail if $k \geq 4$).

I will discuss several generalizations of this result to families of linear patterns in countable abelian groups. The proof strategy involves extending elements of the structural description of characteristic factors for multiple ergodic averages due to Host-Kra and Ziegler to the appropriate setting. In particular, I will explain limit formulas for some classes of multiple ergodic averages controlled by the Conze-Lesigne (quasi-affine) factor in general countable abelian groups. The limit formulas and Khintchine-type recurrence results are new even for $\mathbb{Z}^d$-actions but apply much more generally.

Daniel Altman, University of Oxford, UK.

A non-flag arithmetic regularity lemma and counting lemma

We will discuss a version of the Green–Tao arithmetic regularity lemma and counting lemma which works in the generality of all linear forms. In this talk we will focus on the qualitative and algebraic aspects of the result.

Vitaly Bergelson, Ohio State University, USA.

Ergodic Ramsey theory in the nilpotent setup

We will discuss some old and new open problems which pertain to extending/generalizing some of the known dynamical and combinatorial results involving nilpotent structures.

Sebastian Donoso, University of Chile, Chile.

Joint ergodicity beyond polynomials

In this talk we will present recent joint ergodicity results for various non-polynomial functions of polynomial growth. We will mention the techniques we use, which combine recent works of Frantzikinakis and Tsinas with the machinery of seminorms for multiple averages for polynomial iterates that we have developed in a previous work with A. Ferre-Moragues, A. Koutsogiannis, and W. Sun. If time permits, we will provide comments on open questions and current ongoing work. This is joint work with A. Koutsogiannis and W. Sun.

Tomasz Downarowicz, Wroclaw Univeristy of Science and Technology, Poland.

Multiorders and asymptotic pairs in actions of amenable groups

For topological actions of the integers it is well known that positive topological entropy implies the existence of (forward) asymptotic pairs (see [1]). In the opposite direction we know that an action with zero topological entropy admits an extension which has no asymptotic pairs (see [2]).

Let G be an infinite countable amenable group. Is there a similar characterization of positive vs. zero entropy topological G-actions in terms of appropriately defined asymptotic pairs? During the talk I will address this problem, give a positive answer to the above question and present some necessary tools, in particular provide the notion of asymptoticity that works in this context. For more details, see [3]. This is joint with Mateusz Więcek

References

[1] F. Blanchard, B. Host, S. Ruette, *Asymptotic pairs in positive-entropy systems*, Ergodic Theory Dynam. Systems 22 (2002), no. 3, 671–686.

[2] T. Downarowicz, Y. Lacroix, *Topological entropy zero and asymptotic pairs*, Israel J. Math. 189 (2012), 323–336.

[3] T. Downarowicz, M. Więcek, *Asymptotic pairs in topological actions of amenable groups* https://arxiv.org/abs/2303.12923

Diego González-Sánchez, Alfréd Rényi Institute, Hungary.

Free nilspaces, double-coset nilspaces, and Gowers norms

Compact finite-rank (CFR) nilspaces have become central in the nilspace approach to higher-order Fourier analysis, notably through their role in a general form of the inverse theorem for the Gowers norms. In this talk we discuss these nilspaces per se, and in connection with further refinements of this inverse theorem that were conjectured recently. The first main result states that every CFR nilspace is obtained by taking a *free* nilspace (a nilspace based on an abelian group of the form $\mathbb{Z}^r \times \mathbb{R}^s$) and quotienting this by a discrete group action of a specific type, describable in terms of polynomials. We call these group actions *higher-order lattice actions* as they generalize actions of lattices in $\mathbb{Z}^r \times \mathbb{R}^s$. The second main result (which relies on the first one) represents every CFR nilspace as a *double-coset space* $K\backslash G/\Gamma$ where G is a nilpotent Lie group of a specific kind. These results open the study of these nilspaces to areas more classical than nilspace theory, such as the theory of topological group actions. We discuss briefly the theory of topological *non-compact* nilspaces as we need it in order to prove our results. Our applications include new inverse theorems for Gowers norms that are purely group theoretic in the sense that the correlating harmonics are based on double-coset spaces. This yields progress towards the Jamneshan–Tao conjecture. Joint work with Pablo Candela and Balázs Szegedy.

John Griesmer, Colorado School of Mines, USA.

Separating recurrence properties

A subset S of a countable abelian group G is a set of measurable recurrence if for every subset A of G having positive upper Banach density, the difference set $A - A := \{a - a' : a, a' \in A\}$ has nonempty intersection with S.

S is a set of topological recurrence if for every syndetic subset A of G, the difference set $A - A$ has nonempty intersection with S.

A is a Bohr neighborhood in G if there is a real valued trigonometric polynomial p on G such that $B := \{g \in G : p(g) > 0\}$ is nonempty and A contains B.

S is a set of Bohr recurrence if for every Bohr neighborhood A contained in G, $A-A$ has nonempty intersection with S.

Bohr sets are syndetic, and syndetic sets have positive upper Banach density, so every set of measurable recurrence is a set of topological recurrence, and every set of topological recurrence is a set of Bohr recurrence.

Bergelson asked whether every set of topological recurrence in $\mathbb{Z}$ (the group of integers) is also a set of measurable recurrence. Kriz answered this negatively in 1987. Surprisingly, the question is still open for some other groups, such as the direct sum of countably many copies of $\mathbb{Z}/3\mathbb{Z}$.

Veech asked whether every set of Bohr recurrence is a set of topological recurrence. This question is now known as "Katznelson's problem," and remains open in every countably infinite abelian group.

We will survey recent developments on these problems and related problems. We will also give explicit examples of sets with unknown recurrence properties, i.e. candidate examples which could separate two kinds of recurrence.

Asgar Jamneshan, Koc University, Turkey.

A counterexample to a conjecture of Bergelson, Tao, and Ziegler

Bergelson, Tao, and Ziegler showed that any ergodic $\mathbb{F}_p^\omega$-system (X, μ, T) of order k is an Abramov $\mathbb{F}_p^\omega$-system of order $C(p,k) \geq k$, that is $L^2(X,\mu)$ is generated by the polynomials of degree at most $C(p,k)$. They showed that $C(p,k) = k$ if $k \leq p-1$ and conjectured that $C(p,k) = k$ for all possible values of k, p. Using nilspace methods, Candela, González-Sánchez, and Szegedy showed that the conjecture is true in the slighly wider range $k \leq p+1$. However, Shalom, Tao, and I recently constructed an $\mathbb{F}_2^\omega$-system of order 5 that is not Abramov of order 5, which refutes the conjecture made by Bergelson, Tao, and Ziegler. To do this, we disproved a strong inverse conjecture for the Gowers U^6-norm on $\mathbb{F}_2^n$ using a correspondence principle. We reduced the combinatorial conjecture to a cohomological problem on finite nilspaces, which we disproved using numerical methods. In this talk, I will present the key constructions and verification steps for this counterexample.

Jakub Konieczny, University of Lyon, France.

Automatic sequences from the point of view of higher order Fourier analysis

Automatic sequences, that is, sequences whose n-th term can be computed by a finite automaton given the digits of n as input, have long been studied in number theory, combinatorics, dynamics and computer science. During my talk, I will explore the applications of higher order Fourier analysis to automatic sequences. Together with J. Byszewski and C. Müllner we showed that automatic sequences satisfy a stronger variant of the arithmetic regularity lemma: they can be decomposed into the pseudorandom part, which is Gowers uniform of all orders, and the structured part, which is essentially a 1-step nilsequence. I will discuss the ideas behind this result, as well as some recent applications in upcoming work with B. Adamczewski and C. Müllner on quantitative variants of Cobham's theorem.

Andreas Koutsogiannis, University of Thessaloniki, Greece.

Convergence of multiple ergodic averages for totally ergodic systems

A collection of integer sequences is jointly ergodic if for every ergodic measure preserving system the multiple ergodic averages, with iterates given by the sequences, converge to "the expected limit" in the mean, i.e., the product of the integrals of the functions that appear in the averages. Exploiting a recent approach of Frantzikinakis, which allows one to avoid deep tools from ergodic theory that were previously used to establish similar results, we study the joint ergodicity problem in totally ergodic systems for integer parts of suitable iterates of real polynomial and Hardy field functions of polynomial growth. The motivation for this study, which is joint work with Wenbo Sun, comes from previous work with Dimitris Karageorgos.

Bryna Kra, Northwestern University, USA.

Infinite sumsets and cubic measures

Furstenberg's proof of Szemerédi's Theorem introduced the Correspondence Principle, a general technique for translating a combinatorial problem into a dynamical one. While the original formulation suffices for many patterns, including arithmetic progressions and some infinite configurations, finding infinite subsets required refinements of these tools. Based on joint work with Joel Moreira, Florian Richter, and Donald Robertson, we discuss the translation of infinite sumsets to a dynamical question and show how the cubic measures arise.

Borys Kuca, University of Crete, Greece.

Multiple ergodic averages along polynomials for systems of commuting transformations

Furstenberg's dynamical proof of the Szemerédi theorem initiated a thorough examination of multiple recurrence phenomena; in doing so, it laid the grounds for a new subfield within ergodic theory. Multiple recurrence results are usually deduced from structural characterisations of multiple ergodic averages - analytic objects that generalise the classical Birkhoff averages. Of special interest are averages of commuting transformations with polynomial iterates, which play a central role in the polynomial Szemerédi theorem of Bergelson and Leibman. Their norm convergence has been established in a celebrated paper of Walsh, but for a long time, little more has been known about the form of the limit. A recent outburst of activity on this topic led to new structural results on the limits of such averages, bringing resolution of several previously intractable problems. I will present some of these recent developments with the emphasis on joint works with Nikos Frantzikinakis.

Alejandro Maass, University of Chile, Chile.

On recurrence and nilsystems

In this talk we will revisit some results on recurrence on nilsystems and extensions. In particular, the result saying that Bohr recurrence suffices for recurrence in nilsystems and extensions to multiple recurrence. Then we will see some partial extensions to more general groups actions. This talk is a combination of works with B. Host and B. Kra, and with S. Donoso and F. Hernández.

Freddie Manners, University of California at San Diego, USA.

Cubespaces and homotopy theory

The abstract definition of a *cubespace* or *nilspace*, and the study of these abstract objects, now has a core place in higher order Fourier analysis and ergodic theory. These objects are relatively easy to define, but a large volume of work has been expended understanding their structure.

It has been noted for a while that there are strong analogues between cubespaces and the study of *simplicial sets* or more generally abstract topology and homotopy theory. While the resemblances are striking, it has not been possible to connect the two areas logically, or to tie up cubespace theory with this other well-developed area of mathematics.

In this – rather speculative – talk, I will try to convince you that it is both possible and rewarding to complete this tie-up; i.e., to place existing cubespace theory in a common framework with existing topology and homotopy theory, and exploit the connection.

Luka Milićevic, Mathematical Institute of the Serbian Academy of Sciences and Arts, Serbia.

Approximate quadratic varieties

A classical result in additive combinatorics, which is a combination of Balog-Szemerédi-Gowers theorem and Freiman's theorem says that if a subset A of $\mathbb{F}_p^n$ contains at least $c|A|^3$ additive quadruples, then there exists a subspace V, comparable in size to A, such that $|A \cap V| \geq \Omega_c(|A|)$. Motivated by the fact that higher order approximate algebraic structures play an important role in the theory of uniformity norms, it would be of interest to find higher order analogues of the mentioned result. In this talk, I will discuss a quadratic version of the approximate property in question, namely what it means for a set to be an approximate quadratic variety. I will also say something about structure of such a set.

Joel Moreira, University of Warwick, UK.

Infinite arithmetic configurations in sets with positive density

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Seeking a density version for Hindman's finite sums theorem, Erdos asked several questions in the 1970's about what kind of infinite structures can be found in every set of natural numbers with positive density. I will present recent joint work with Kra, Richter and Robertson where we answered some of these questions, and discuss some related questions.

Sarah Peluse, University of Michigan, USA.

Bounds for sets without L-shapes

I will discuss the difficult problem of proving reasonable bounds in the multidimensional Szemerédi theorem and describe a proof of such bounds for sets lacking nontrivial "L-shapes", i.e., the configuration $(x, y), (x, y + z), (x, y + 2z), (x + z, y)$.

Donald Robertson, University of Manchester, UK.

Disintegrating cubic measures for sumsets

Cubic measures are important in many combinatorial applications of ergodic theory. In this talk we describe how standard disintegrations of cubic measures can be refined, and how these refinements contribute to the existence of sumsets in sets of positive density. Based on joint work with Bryna Kra, Joel Moreira and Florian Richter.

Florian Richter, EFPL, Switzerland.

Uniformity norms and Hindman's conjecture

In this talk we will prove that the configuration $\{x+y, xy\}$ is controlled by the local Host-Kra uniformity norms of step 2. This will allow us to derive a density analogue of (a special case of) a theorem of Moreira.

Or Shalom, IAS, USA.

Inverse theorems in ergodic theory and additive combinatorics

The Gowers uniformity k-norm on a finite abelian group measures the averages of complex functions on such groups over k-dimensional arithmetic cubes. The inverse question about these norms asks if a large norm implies correlation with a function of an algebraic origin. The analogue of the Gowers uniformity norms for measure-preserving abelian actions are the Host-Kra-Gowers seminorms which are intimately connected to the Host-Kra-Ziegler factors of such systems. The corresponding inverse question in the dynamical setting asks for a description of such factors in terms of systems of an algebraic origin. In this talk, we survey recent results about the inverse question in the dynamical and combinatorial settings, and in particular how an answer in the former setting can imply one in the latter. This talk is based on joint works with Asgar Jamneshan and Terence Tao.

Wenbo Sun, Virginia Tech, USA.

Modern tools in the joint ergodicity problem

It is well known by the Mean Ergodic Theorem that for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and L^∞ function f, the time average of $T^n f$ converges to the integral of f if and only if T is ergodic. It is a natural question to ask when the average of products of polynomial iterates of L^∞ functions (known as multiple ergodic averages) converges to the product of the integrals of the functions. This question is called the Joint Ergodicity Problem. In this talk, I will introduce some recent advances in this problem using modern tools in ergodic theory, which includes the structure theorem, the concatenation theorem and Frantzikinakis' criterion. This talk is based on joint works with Sebastián Donoso, Andreu Ferré Moragues and Andreas Koutsogiannis.

Joni Teräväinen, University of Turku, Finland.

On Elliott's conjecture and applications

About 30 years ago, Elliott posed the conjecture that a bounded multiplicative function should not correlate with its own shifts, unless the function is "close to" a twisted Dirichlet character. This conjecture includes Chowla's conjecture as a special case. I will discuss a recent work where we prove a version of this conjecture in some new cases, improving on previous work of Tao and I. I will also discuss applications of these results to e.g. Furstenberg systems and sign patterns of multiplicative functions. This is joint work with O. Klurman and A. P. Mangerel.

Matthew Tointon, University of Bristol, UK.

Approximation of vertex-transitive graphs by nilpotent groups, and applications to probability

Classical theorems of Gromov and Trofimov imply that an arbitrary vertex-transitive graph of polynomial growth can be approximated in a certain algebraic sense by a virtually nilpotent group. This fact has in turn had important applications to the probabilistic properties of vertex-transitive graphs, in particular the recurrence or transience of random walks, and the existence of percolation phase transitions. I will report on a finitary analogue of the Gromov-Trofimov theory, initiated by Breuillard, Green and Tao in the Cayley-graph case and refined and generalised to arbitrary vertex-transitive graphs by Tessera and me. I will then describe finitary versions of the aforementioned probabilistic applications, joint with Tessera and with Hutchcroft.

Konstantinos Tsinas, University of Crete, Greece.

Multiple ergodic theorems along sequences of polynomial growth

Following the classical results of Host-Kra and Leibman on the polynomial ergodic theorem, it is natural to ask whether a similar mean convergence result holds along several other sequences of polynomial growth. In 1994, Boshernitzan gave necessary and sufficient conditions under which the sequence $f(n)$ is equidistributed on $\mathbb{T}$, where f belongs to a large class of smooth functions (called a Hardy field) and which has polynomial growth. In conjunction with the spectral theorem, this implies a corresponding mean convergence theorem in the case of single ergodic averages along the sequence $\lfloor f(n) \rfloor$ of integer parts.

We investigate the same question in the general setting of multiple ergodic averages. We establish necessary and sufficient conditions under which these averages converge to the product of the integrals in ergodic systems (the jointly ergodic case). Using an appropriate equidistribution result on nilmanifolds, we also prove a mean convergence result in the "linearly dependent" case. The main ingredients are the recent results of Frantzikinakis on joint ergodicity of sequences and a new method for handling averages of Hardy field functions in short intervals, which is used to establish the required seminorm estimates and equidistribution results on nilmanifolds.

Máté Wierdl, University of Memphis, USA.

Distribution of translation orbits along good averaging sequences

Let $S := \{s_1 < s_2 < \dots\}$ be a strictly increasing sequence of positive integers so that in every dynamical system (X, μ, T) and $f \in L^2(X)$, the sequence $\left(\frac{1}{N}\sum_{n\le N} f(T^{s_n}x)\right)_{N\in\mathbb{N}}$ of ergodic averages along (s_n) converge in L^2-norm. What can be the limit? We know that in case $s_n = n$ and an ergodic T the limit is $\int_X f\,d\mu$. In case $s_n = n^2$ or when s_n is the nth prime number, the limit is again $\int_X f\,d\mu$ provided $T$is totally ergodic, that is, T^k is ergodic for every positive integer k. By the spectral theorem, the limit being the integral in totally ergodic systems is equivalent with $\lim_N \frac{1}{N}\sum_{n\le N} e(s_n\alpha) = 0$ for every irrational α, where we used Weyl's notation $e(\beta) := e^{2\pi i\beta}$. This limit being 0 is equivalent with saying that the limit Borel probability measure $\mu_{S,\alpha} := \lim_N \frac{1}{N}\sum_{n\le N} \delta_{s_n\alpha}$ for irrational α is the Haar-Lebesgue measure on the torus $\mathbb{T} := [0, 1)$.

Can this limit measure be other Borel probability measures? It turns out that $\mu_{S,\alpha}$ must be a continuous measure if α is irrational. Our main question hence is:

Can the limit measure $\mu_{S,\alpha}$ be any continuous measure for irrational α?

Well, if the answer was yes, then it would confirm Furstenberg's $\times 2 \times 3$ conjecture. But no, we do not know the answer to this question, though we have been trying for a while now. We have figured out quite a few things, though. We know, for example, that the answer will depend on the irrational α, and along the way we obtain a new characterization of Rajchman measures, that is, measures with vanishing Fourier coefficients at infinity.

In the talk, we will see that besides our main question, there are many other questions, not to mention extending the problem and existing results to other compact groups and nilmanifolds.

XiangDong Ye, University of Science and Technology of China, China.

A generalization of Furstenberg-Glasner's result to polynomials

It is proved that if $S \subset \mathbb{Z}$ is piecewise syndetic, $d \in \mathbb{N}$ and $p_1, \dots, p_d$ are integral polynomials vanishing at 0, then $\{(m, n) \in \mathbb{Z}^2 : m+p_1(n) \in S, \dots, m+p_d(n) \in S$ is piecewise syndetic in $\mathbb{Z}^2$, which extends Furstenberg-Glasner's result for linear polynomials to general integral polynomials.