# On recurrence and nilsystems

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Many classical connections between ergodic theory and combinatorics is via recurrence properties of dynamical systems.

- Van der Waerden: any finite coloring of the integers contains arbitrary long monochromatic arithmetic progressions.
- Many refinements have to do with the restrictions a "recurrence set" has.
- As was stated many times, the relation between recurrence and nilsystems has become a natural direction to study.

# Recurrence for actions of $\mathbb{Z}$

- Let $(X, T)$ be a minimal system ($X$ compact metric, $T : X \to X$ homeomorphism s.t. all orbits are dense).
- $R \subseteq \mathbb{N}$ is a set of (topological) recurrence if for every minimal system $(X, T)$ and every nonempty open set $U \subseteq X$,

$$\exists\, n \in R,\ U \cap T^{-n}U \neq \emptyset$$

- Let $s \geq 1$. $R \subseteq \mathbb{N}$ is a set of $s$-recurrence if for every minimal system $(X, T)$ and every nonempty open set $U \subseteq X$,

$$\exists\, n \in R,\ U \cap T^{-n}U \cap T^{-2n}U \ldots \cap T^{-ns}U \neq \emptyset$$

- $R \subseteq \mathbb{N}$ is of multiple recurrence if it is of $s$-recurrence for every $s \geq 1$.

# Recurrence for actions of $\mathbb{Z}$

- $S \subseteq \mathbb{N}$ infinite, $S - S$ is a set of recurrence (Furstenberg).
- Sets of multiple recurrence: IP-sets, shifted primes, ....
- Examples where simple recurrence is different from multiple recurrence.
- Examples of sets of 1-recurrence and not 2-recurrent (Furstenberg).
- Examples of sets of $s$-recurrence but not $s+1$-recurrence (Frantzikinakis, Lesigne and Wierdl).
- More general constructions by Ye, Huang and Song using generalized regionally proximal relations.

# Recurrence sets for families of systems

Let $\mathcal{F}$ be a family of systems (rotations on compact Abelian groups, distal systems, $s$-nilsystems, ...).

- A subset $R \subseteq \mathbb{N}$ is a set of (topological) recurrence for the family of systems $\mathcal{F}$ if for every minimal system $(X, T)$ in $\mathcal{F}$ and every nonempty open set $U \subseteq X$,

$$\exists\, n \in R, \ U \cap T^{-n}U \neq \emptyset$$

- Let $s \in \mathbb{N}$. A subset $R \subseteq \mathbb{N}$ is a set of $s$-recurrence for the family of systems $\mathcal{F}$ if for every minimal system $(X, T)$ in $\mathcal{F}$ and every nonempty open set $U \subseteq X$,

$$\exists\, n \in R, \ U \cap T^{-n}U \cap T^{-2n}U \ldots \cap T^{-sn}U \neq \emptyset$$

# Bohr Recurrence and $G = \mathbb{Z}$

Starting from simple systems:

- A set of recurrence for minimal translations on compact abelian groups is called a set of Bohr recurrence.
- Thus, $R \subseteq \mathbb{N}$ is a set of Bohr recurrence if for $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in \mathbb{T}$ and $\epsilon > 0$, there exists $n \in R$ such that $||\alpha_1 n|| < \epsilon, \ldots, ||\alpha_k n|| < \epsilon$.

A very studied/celebrated question:

**Question (Katznelson)**

*Is a set of Bohr recurrence of recurrence?*

# Strategy via extensions

- A classical approach is to use "transfinite inductions" for special type of extensions and see if recurrence sets can be lifted by such extension. Recall: $(X, T)$ is an extension of $(Y, S)$ if there is an onto continuous map $\pi : X \to Y$ s.t. $\pi \circ T = S \circ \pi$.
  In several important results of topological dynamics this strategy has worked.
- It turns out that: $s$-recurrence can be lifted by proximal extensions (points $x, y$ in the same fiber for $\pi$ are proximal, i.e., $\inf_{n \geq 0} d(T^n x, T^n y) = 0$).
- But it is not known if $s$-recurrence can be lifted by isometric, equicontinuous or distal extensions. Nice progress comes from Glasscock, Koutsogiannis and Richter (2021), who proved that sets of Bohr recurrence are sets of recurrence for skew product extensions of an equicontinuous system by a $d$-dimensional torus.

# Strategy via extensions

The following diagram summarizes this strategy.

$$\begin{tikzcd}
(Y,\mathbb{Z}) \arrow[r, "\text{weakly mixing}"'] \arrow[dr, "\text{proximal}"'] & (Y_{PI},\mathbb{Z}) \arrow[r, "\text{proximal}"'] & \cdots \arrow[r, "\text{equicontinuous}"'] & (Y_2,\mathbb{Z}) \arrow[r, "\text{proximal}"'] & (Y_1,\mathbb{Z}) \arrow[r, "\text{equicontinuous}"'] & \boxed{(\{1\},\mathbb{Z})} \\
 & (X,\mathbb{Z}) & & & &
\end{tikzcd}$$

# Strategy via extensions

Another strategy concerning factors has to do with maximal pronilfactors associated to a system.

$$(X,\mathbb{Z}) \xrightarrow[\text{weakly mixing}]{} (X_d,\mathbb{Z}) \xrightarrow[\text{distal}]{} \boxed{(\mathcal{Z}_\infty,\mathbb{Z})} \xrightarrow[\text{inverse limit}]{} \cdots \xrightarrow[\text{g.e.}]{} \boxed{(\mathcal{Z}_2,\mathbb{Z})} \xrightarrow[\text{g.e.}]{} \boxed{(\mathcal{Z}_1,\mathbb{Z})}$$

- In this case, the strategy is to lift Bohr recurrence through the finite pronilfactors first, and then to the maximal $\infty$-step pronilfactor.
- We know from [Dong-Donoso-M-Song-Ye] that any minimal system without nontrivial pairs with arbitrarily long finite IP-independence sets ($\{x,y\}^{[d]} \subseteq Q^d(X,T)$ for al $d \geq 1$) is an almost one-to-one extension of its maximal $\infty$-step nilfactor and is uniquely ergodic.

A first progress in this strategy is given by:

**Theorem (Host-Kra-M)**

*Let $R \subseteq \mathbb{N}$ be aBohr set, then it is a set of recurrence for all s-nilsystems.*

- As Bohr recurrence can be lift through $\mathbb{Z}$-inverse limits, this result proved that Bohr recurrence can be lifted up to $\infty$-step $\mathbb{Z}$-pronilsystems.
- This is also true for proximal extensions of previous systems.

# Sketch of the Proof

- The proof of this result has two flavours, there is a purely topological proof, which uses in a delicate manner the nil structure, and other, which putting additional hypotheses or for $s = 2$ is quasi purely measure theoretical.
- I sketch first the one with the extra hypothesis: $(X = G/\Gamma, T)$ is a minimal $s$-step nilsystem with $G$ connected as well as for $s = 2$.

# Sketch of the Proof

- Under these hypotheses we proved in a previous work for $s = 2$ and was proved before when $G$ is connected that for $f \in L^2(\mu)$ s.t. $\mathbb{E}(f|Z) = 0$ one has:

$$\int_X T^n f \cdot \overline{f}\, d\mu \text{ goes to } 0 \text{ as } n \to +\infty$$

- Let $R \subseteq \mathbb{N}$ be a Borh set and $U \subseteq X$ an open nonempty set. Clearly, minimality gives, $\mu(U) > 0$.
- Consider $g = \mathbb{E}(1_U|Z)$ and $f = 1_U - g$. So,

$$1 \geq ||g||_{L^2(m_Z)} \geq ||g||_{L^1(m_Z)} = \mu(U)$$

- Also, from $\mathbb{E}(f \mid Z) = 0$, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ s.t.

$$\left| \int_X T^n f \cdot f\, d\mu \right| < \epsilon \text{ for every } n \geq n_0$$

# Sketch of the Proof

- In the Kronecker factor $(Z, S)$ we have that for some $\delta > 0$ if $d_Z(t, 0) < \delta$ then $||g(\cdot + t) - g||_{L^2(m_Z)} < \epsilon$.
- In particular, consider $n \in R$ with $n \geq n_0$ and $d_Z(n\alpha, 0) < \delta$ (here $\alpha$ is the rotation in $Z$).
- We get,

$$\begin{aligned}\mu(U \cap T^{-n}U) =& \int_X f \cdot T^n f \, d\mu + \int_Z g \cdot S^n g \, dm_Z \\ \geq& -\epsilon + ||g||^2_{L^2(m_Z)} - ||g||_{L^2(m_Z)} ||S^n g - g||_{L^2(m_Z)} \\ \geq& \mu(U)^2 - 2\epsilon \geq \mu(U)^2/2 > 0,\end{aligned}$$

taking $\epsilon = \mu(U)^2/4$.

# Sketch of the Proof

The general proof is by induction on $s \geq 2$ after reducing to the case where $X$ is connected and $G_0$ simply connected.

- Define
$$\tilde{G} = G/G_s, \ \tilde{\Gamma} = \Gamma/(\Gamma \cap G_s) \ \text{ and } \tilde{X} = \tilde{G}/\tilde{\Gamma}.$$
- Then $\tilde{X}$ is an $(s-1)$-step nilmanifold, and the quotient map $G \to \tilde{G}$ induces a projection $\pi\colon X \to \tilde{X}$. Thus we can view $\tilde{X}$ as the quotient of $X$ under the action of $G_s$.
- Let $\tilde{\tau}$ be the image of $\tau$ in $\tilde{G}$ and $\tilde{T}$ be the translation by $\tilde{\tau}$ on $\tilde{X}$. Then $(\tilde{X}, \tilde{\tau})$ is an $(s-1)$-step nilsystem and $\pi\colon X \to \tilde{X}$ is a factor map.

# Sketch of the Proof

- From the induction hypothesis there exist arbitrarily large $n \in R$ with

$$\pi^{-1}(B(e_{\tilde{X}}, \epsilon)) \cap \tilde{T}^{-n}\pi^{-1}(B(e_{\tilde{X}}, \epsilon)) \neq \emptyset$$

- For these values of $n$, there exist $x \in X$ and $v \in G_s$ with $d_X(x, e_X) < \epsilon$ and $d_X(T^n x, v \cdot e_X) < \epsilon$. Lifting $x$ to $G$, we obtain $g \in G$ and $\gamma \in \Gamma$ with

$$d_G(g, 1_G) < \epsilon \text{ and } d_G(\tau^n g, v\gamma) < \epsilon$$

## Sketch of the Proof

- Main Claim: for $n$ sufficiently large there exists $h \in G_{s-1}$ and $\theta \in G_s \cap \Gamma$ such that

$$d_G(h, 1_G) < \epsilon \text{ and } d_G([h^{-1}, \tau^n], v^{-1}\theta) < \epsilon.$$

- From here, write $y = h \cdot x$, which is the projection of $hg$ in $X$ and

$$d_X(y, e_X) \leq d_G(hg, 1_G) \leq d_G(h, 1_G) + d_G(g, 1_G) < 2\epsilon$$

- Furthermore,

$$\begin{aligned} d_X(T^n y, e_X) &\leq d_G(\tau^n hg, \theta\gamma) = d_G(h[h^{-1}, \tau^n]\tau^n g, \theta\gamma) \\ &\leq \epsilon + d_G([h^{-1}, \tau^n]\tau^n g, \theta\gamma) = \epsilon + d_G(\tau^n g[h^{-1}, \tau^n], \theta\gamma) \\ &\leq 2\epsilon + d_G(v\gamma[h^{-1}, \tau^n], \theta\gamma) = 2\epsilon + d_G([h^{-1}, \tau^n]v\gamma, \theta\gamma) \\ &= 2\epsilon + d_G([h^{-1}, \tau^n], v^{-1}\theta) < 3\epsilon, \end{aligned}$$

where we used the fact that $[h^{-1}, \tau^n] \in G_s$ and that $G_s$ is included in the center of $G$.

# $s$-Recurrence sets for $G = \mathbb{Z}$

Some definitions from Host and Kra work:

- Let $s \geq 1$ be an integer. The set $E \subseteq \mathbb{N}$ is a $\mathrm{Nil}_s\text{-}\mathrm{Bohr}_0$ set if there exist a minimal $s$-step nilsystem $(X, T)$, $x_0 \in X$ and an open neighborhood $U \subseteq X$ of $x_0$ such that

$$\{n \in \mathbb{N} \colon T^n x_0 \in U\} \subseteq E$$

- A set $R \subseteq \mathbb{N}$ is a $\mathrm{Nil}_s\text{-}\mathrm{Bohr}_0^*$ set if it has nonempty intersection with all $\mathrm{Nil}_s\text{-}\mathrm{Bohr}_0$ sets.
- Huang, Shao and Ye proved: If $E \subseteq \mathbb{N}$ is a $\mathrm{Nil}_s\text{-}\mathrm{Bohr}_0$ set, then there exist a minimal $s$-step nilsystem $(X, T)$ and a nonempty open set $U \subseteq X$ such that

$$E \supseteq N^s(U) = \{n \in \mathbb{N} : U \cap T^{-n} U \cap T^{-2n} U \ldots \cap T^{-sn} U \neq \emptyset\}$$

## $s$-Recurrence sets for $G = \mathbb{Z}$

### Proposition

*Let $s \geq 1$. For $R \subseteq \mathbb{N}$, the following are equivalent:*

1. *$R$ is a set of $s$-recurrence for minimal $s$-step nilsystems*
2. *$R$ is a set of pointwise recurrence for minimal $s$-step nilsystems*
3. *$R$ is a $\mathrm{Nil}_s\text{-}\mathrm{Bohr}_0^*$ set*

Some consequences: let $s, \ell \geq 1$ be integers and let $R \subseteq \mathbb{N}$.

- If $s \leq \ell$, $R$ is a set of $\ell$-recurrence for minimal $s$-step nilsystems if and only if it is a set of $s$-recurrence for minimal $s$-step nilsystems if and only if it is a $\mathrm{Nil}_s\text{-}\mathrm{Bohr}_0^*$ set.
- However, when $s > \ell$, other than for $\ell = 1$, the following statement is not direct: if $R$ is a set of $\ell$-recurrence for $s$-step nilsystems, then $R$ is a set of $\ell$-recurrence for all $t$-step nilsystems for any $t \geq s$.

# $s$-Recurrence sets for $G = \mathbb{Z}$

**Theorem (Host-Kra-M)**

*Let $R \subseteq \mathbb{N}$ be a s-recurrent set for affine s-nilsystems. Then, for all $t \geq s$ it is an s-recurrent set for affine t-nilsystems.*

Recall: let $M$ be a $s \times s$ *unipotent* matrix with integer entries (i.e., $(M - I)^s = 0$) and $\alpha \in \mathbb{T}^s$. Define $T \colon \mathbb{T}^s \to \mathbb{T}^s$ by $T(x) = Mx + \alpha$ (operations are $\mod 1$). The system $(\mathbb{T}^s, T)$ is called an *affine system on* $\mathbb{T}^s$. It is minimal if the projection of $\alpha$ on $\mathbb{T}^s / \ker(M - I)$ generates a minimal rotation on this torus. The system $(\mathbb{T}^s, T)$ can be represented as a $s$-nilsystem.

Previous results were generalized in some unpublished notes together with Wembo Sun:

**Theorem**

*Let $R \subseteq \mathbb{N}$ be a s-recurrent set for s-nilsystems. Then, for all $t \geq s$ it is an $s$-recurrent set for $t$-nilsystems.*

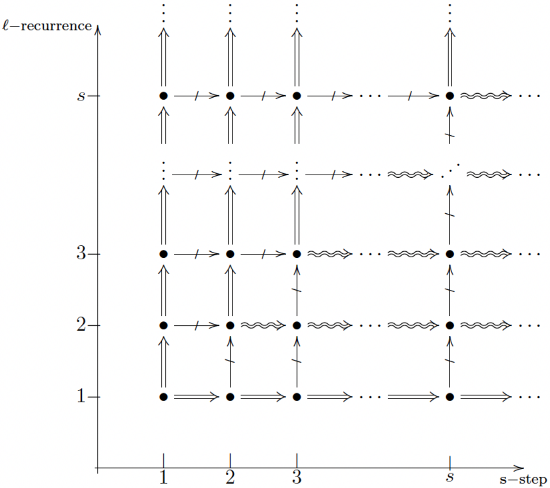
## $s$-Recurrence sets for $G = \mathbb{Z}$

Finally, I want to mention the following very interesting result, which is a powerful tool to construct many types of examples and counter examples. It uses the machinery of $s$-regionally proximal relations:

**Theorem**

*Let $(X, T)$ be a minimal $s$-step nilsystem and let $(x_0, x_1) \in RP^{[s-1]}(X, T)$. Let $U$ be an open neighborhood of $x_1$ with $x_0 \notin \overline{U}$. Then $N(x_0, U)$ is a set of $(s-1)$-recurrence, is not a set of $s$-recurrence (even for $s$-step nilsystems) and is not a set of pointwise recurrence.*

# Summarizing



# Recurrence and general group actions

- Let $G$ be a locally compact abelian group and assume it acts on the compact metric space $X$ by homeomorphisms. Denote $(X, G)$ such dynamical system.
- $R \subseteq G$ is a set of recurrence for $(X, G)$ if $\forall U \subseteq X$ a nonempty open set

$$R \cap \{g \in G \mid U \cap g^{-1}U \neq \emptyset\} \neq \emptyset.$$

- Bohr recurrence sets are defined, as before, for the family of rotations. So, Katznelson question can be asked in this context: Bohr recurrence sets are of recurrence.

# Recurrence for $G = \mathbb{Z}^d$

We explore different cases in where sets of Bohr recurrence are sets of recurrence for $\mathbb{Z}^d$-nilsystems. The first one needs some particular cube structures (Cabezas-Donoso-M):

- Let $(X, T_1, \cdots, T_d)$ be a $\mathbb{Z}^d$-system. The set of directional dynamical cubes associated to $(X, T_1, \cdots, T_d)$ is defined by

$$Q_{T_1,\cdots,T_d}(X) = \overline{\{(T_1^{n_1\epsilon_1}\ldots T_d^{n_d\epsilon_d}x)_{\epsilon\in\{0,1\}^d} : x \in X, \vec{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d\}} \subseteq X^{[d]}$$

- $(X, T_1, \cdots, T_d)$ has the unique closing property if whenever $x, y \in Q_{T_1,\cdots,T_d}(X)$ have $2^d - 1$ coordinates in common then $x = y$.

Using this notion, we have that:

**Theorem (Donoso-Henández-M)**

*Let $R \subseteq \mathbb{Z}^2$ a set of Bohr recurrence. Then $R$ is a set of recurrence for all $\mathbb{Z}^2$-minimal nilsystems with the unique closing parallelepiped property.*

To extend this result we need to introduce a stronger notion of closing property.

**Theorem (Donoso-Henández-M)**

*Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence. Then $R$ is a set of recurrence for all $\mathbb{Z}^d$-minimal nilsystems with the "strong" unique completeness property.*

# Recurrence for $G = \mathbb{Z}^d$, quasi affine case

When $G = \mathbb{Z}^d$ we can consider the class of affine nilsystems. Here $X = \mathbb{T}^r$ and the transformations $T_1, \cdots, T_d$ are defined by $T_i(x) = A_i x + \alpha_i$ commutes, where the matrices $(A_i)_{i=1}^d$ are unipotent and commute as well.
Frantzikinakis and Kra showed that $\mathbb{Z}^d$-affine nilsystems can be characterized by the fact that $G_0$ is abelian and

**Proposition**

*Let $X = G/\Gamma$ be a connected nilmanifold such that $G_0$ is abelian. Then any nilrotation $T_\tau(x) = \tau x$ defined on $X$ with the Haar measure $\mu$ is isomorphic to a unipotent affine transformation on some finite dimensional torus.*

- This inspire the following definition: a $s$-step $\mathbb{Z}^d$-nilsystem is quasi-affine if $G_0$ is abelian.
- It can be proved that, $\mathbb{Z}^d$-quasi-affine nilsystems are basically finite union of $\mathbb{Z}^d$-affine nilsystems.

## Recurrence for $G = \mathbb{Z}^d$, quasi affine case

We have the following extension of Host-Kra-M-Sun result:

**Theorem (Donoso-Henández-M)**

*Let $R \subseteq \mathbb{Z}^d$ be a set of Bohr recurrence. Then for every integer $s \geq 1$, $R$ is a set of recurrence for every minimal $s$-step $\mathbb{Z}^d$-quasi-affine nilsystem.*

# Recurrence for $G = \mathbb{Z}^d$, quasi affine case

The proof needs a reduction to particular recurrence sets. Let $R \subseteq \mathbb{Z}^d$ be an essential set of Bohr recurrence

- Notion of Bohr correlations: $P := (P_{i,j})_{j \geq i} \in [0,1]^{d \times (d+1)/2}$ such that $\forall \epsilon > 0$

  $$R_{P,\epsilon} := \{n \in R \ : \ |\frac{n_j}{n_i} - P_{i,j}| \leq \epsilon, \ \forall j \geq i\},$$

  is a set of Bohr recurrence.
- $\mathcal{BC}(R) \subseteq \mathbb{R}^{d \times (d+1)/2}$, the set of Bohr correlations of $R$, is nonempty.
- Notion of complete independence: a Bohr correlation $P \in \mathcal{BC}(R)$ has complete independence if for all $i \in [d]$,

  $$\{P_{i,j} \mid j \geq i, P_{i,j} \neq 0\},$$

  are rationally independent.

Thanks !!