Convergence of multiple ergodic averages for totally ergodic systems

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$(a_i(n))_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ are "appropriate" integer-valued sequences.

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- T is totally ergodic iff for all $W \in \mathbb{N}$, $r \in \mathbb{Z}$ and $f \in L^2$ we have (in L^2-norm) that

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A convergence result

Theorem (Furstenberg, 1977)

If $(X, \mathcal{B}, \mu, T)$ *is a w.m. system, then for every* $k \in \mathbb{N}$ *and* $f_1, \ldots, f_k \in L^\infty$ *we have*

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k \to \prod_{i=1}^{k} \int_X f_i \, d\mu,$$

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as $N \to \infty$, where the convergence takes place in L^2.

By studying the aforementioned averages, Furstenberg showed (and generalized) Szemerédi's theorem on arithmetic progressions in subsets of natural numbers of positive upper density, stating it as a recurrence problem.

A suitable class of functions: Polynomials

The non-constant integer polynomials $\{p_1, \ldots, p_k\}$ (i.e., $p_i \in \mathbb{Q}[x]$ with $p_i(\mathbb{Z}) \subseteq \mathbb{Z}$) are called *essentially distinct* if $p_i - p_j \not\equiv$ constant $\forall\ i \neq j$.

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The study along integer polynomial iterates led to (multidimensional) polynomial extensions of Szemerédi's theorem (Bergelson-Leibman, 1996).

Crucial tool – van der Corput trick

Lemma (van der Corput, 1931, Bergelson, 1986)

Suppose that $(x_n)_n$ *is a bounded sequence in a Hilbert space and suppose that for any* $h \geqslant h_0 > 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_n, x_{n+h} \rangle = 0,$$

then

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The expression $\langle x_n, x_{n+h} \rangle$ leads to “derivatives” (differences) and reduction of the complexity of the sequences (**PET** induction). (This is the ONLY tool that we have for reduction of complexity in all cases.)

Convergence for polynomial iterates

Conjecture (Bergelson-Leibman, 1996)

Let $d, k \in \mathbb{N}$, $(X, \mathcal{B}, \mu, T_1, \ldots, T_d)$ be a system with commuting T_i's. Then, for every integer polynomials $p_{i,j}$ and $f_j \in L^\infty$, the expression

$$\frac{1}{N}\sum_{n=1}^{N}\left(\prod_{i=1}^{d} T_i^{p_{i,1}(n)}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{d} T_i^{p_{i,k}(n)}\right) f_k$$

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Problem: Find the limit! (The previous results provide no info on it.)

Joint ergodicity

For a collection of sequences $a_1, \ldots, a_k : \mathbb{N} \to \mathbb{Z}$, and a system $(X, \mathcal{B}, \mu, T)$, we say that $(a_1(n))_n, \ldots, (a_k(n))_n$ are

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  For $k = 1$, we say that $(a_1(n))_n$ is *totally ergodic*. If we have it for $r = 0$ and some $W \in \mathbb{N}$, we say that we have *W-joint ergodicity*.

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Problem: Weaken the assumptions on the system, strengthening simultaneously the assumptions on the iterates (OR, do the opposite!), and get (total) joint ergodicity results.

Independent sequences of polynomials

Let $V \subseteq \mathbb{R}$ with $0 \in V$ and $p_1, \ldots, p_k \in \mathbb{R}[x]$. We say that $p_1, \ldots, p_k$ are *V-independent* if $\lambda_1 p_1 + \cdots + \lambda_k p_k \in \mathbb{Q}[x] + \mathbb{R}, \lambda_i \in V, 1 \leqslant i \leqslant k$, implies that $\lambda_1 = \cdots = \lambda_k = 0$;

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($p_1, \dots, p_k$ V-independent means that any non-trivial linear combination of the p_i's with scalars from V has at least one non-constant irrational coefficient.)

More results

Theorem (Frantzikinakis-Kra, 2005)

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Remark

The theorem is actually a characterization.

Convergence to the expected limit for a general system

Theorem (Frantzikinakis, 2015)

Let $p \in \mathbb{R}[x]$ with $p(x) \neq cq(x) + d$, where $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[x]$ (i.e., p is $\mathbb{R}$-independent). Then for every $k \in \mathbb{N}$, any system $(X, \mathcal{B}, \mu, T)$ and $f_1, \ldots, f_k \in L^\infty$, we have that the expression

$$\frac{1}{N}\sum_{n=1}^{N} T^{[p(n)]}f_1 \cdot T^{2[p(n)]}f_2 \cdot \ldots \cdot T^{k[p(n)]}f_k$$

has the same limit (in L^2) as $N \to \infty$ with

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With this result we are getting a refinement of Szemerédi's theorem.

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Theorem (Karageorgos-K., 2017)

For $k \in \mathbb{N}$ let $p_1, \ldots, p_k$ be $\mathbb{R}$-independent real polynomials. Then $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ are (totally) jointly ergodic for all ergodic systems.

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• Show an equidistribution property on nilmanifolds.

More details on the condition of the iterates

• (Similar to Frantzikinakis' result for Hardy field functions, 2009) Let $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathbb{R}[x]$ be $\mathbb{R}\backslash\mathbb{Q}_*$-independent. Let $X = G/\Gamma$ be a nilmanifold with G connected and simply connected, and elements $b_i \in G$ acting ergodically on X. Then the sequence $(b_1^{p_1(n)}\Gamma, \ldots, b_k^{p_k(n)}\Gamma)_n$ is equidistributed in the nilmanifold X^k.

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• So, we want to have that, for all $s \in \mathbb{R}_*$, $p_1/s, \ldots, p_k/s$ are $\mathbb{R}\backslash\mathbb{Q}_*$-independent, or, equivalently, that $p_1, \ldots, p_k$ are $\mathbb{R}$-independent.

Two conjectures for totally ergodic systems

Following the philosophy "weaken the assumptions of the iterates, strengthen the assumptions on the system", one may assume that with "a lot" of ergodicity we have the following.

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Conjecture 2

Conjecture 1 is a characterization.

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where $e(t) := e^{2\pi i t}$, and

$$\text{Spec}(T) := \big\{ t \in [0, 1) : \ Tf = e(t)f \text{ for some nonzero } f \in L^2(\mu) \big\}.$$

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Theorem (Frantzikinakis, 2021)

For $k \in \mathbb{N}$, let $a_1, \ldots, a_k : \mathbb{N} \to \mathbb{Z}$ be sequences. $(a_1(n))_n, \ldots, (a_k(n))_n$ are jointly ergodic for an ergodic system $(X, \mathcal{B}, \mu, T)$ iff $(a_1(n), \ldots, a_k(n))_n$ is good for seminorm estimates and equidistribution for $(X, \mathcal{B}, \mu, T)$.

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Let $p_1, \ldots, p_k \in \mathbb{Q}[x] + \mathbb{R}$. $p_1, \ldots, p_k$ are $\mathbb{R}\backslash\mathbb{Q}_*$-independent iff $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ are totally jointly ergodic for every totally ergodic system.

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Let $p_1, \ldots, p_k \in \mathbb{R}[x]$ so that all the irrational polynomials in $p_1, \ldots, p_k$, if any, are $\mathbb{Q}$-independent. $p_1, \ldots, p_k$ are $\mathbb{R}\backslash\mathbb{Q}_*$-independent iff $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ are totally jointly ergodic for every totally ergodic system.

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Theorem (K.-Sun, 2023)

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Idea: Twisting the argument of Frantzikinakis.

Special cases

Theorem (K.-Sun, 2023)

Let $k \in \mathbb{N}$, $(X, \mathcal{B}, \mu, T)$ be a totally ergodic system, and $p_1, \ldots, p_k \in \mathbb{Q}[x] + \mathbb{R}$ (resp. $p_1, \ldots, p_k \in \mathbb{R}[x]$ so that all the irrational polynomials in $p_1, \ldots, p_k$, if any, are $\mathbb{Q}$-independent). Then the following are equivalent:

(i) $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ *are totally jointly ergodic for* $(X, \mathcal{B}, \mu, T)$.

(ii) *There exists* $W_0 \equiv W_0(p_1, \ldots, p_k) \in \mathbb{N}$ *such that* $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ *are* $W!$*-jointly ergodic for* $(X, \mathcal{B}, \mu, T)$ *for all* $W \geqslant W_0$.

(iii) *There exists an infinite set* $I \equiv I(p_1, \ldots, p_k) \subseteq \mathbb{N}$ *such that* $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ *are* $W!$*-jointly ergodic for* $(X, \mathcal{B}, \mu, T)$ *for all* $W \in I$.

(iv) $p_1, \ldots, p_k$ *are* $S(T)$*-independent (resp.* $S(T) \setminus \mathbb{Z}_*$*-independent).*

Special cases

Theorem (K.-Sun, 2023)

Let $k \in \mathbb{N}$, $(X, \mathcal{B}, \mu, T)$ be a totally ergodic system, and $p_1, \ldots, p_k \in \mathbb{Q}[x] + \mathbb{R}$ (resp. $p_1, \ldots, p_k \in \mathbb{R}[x]$ so that all the irrational polynomials in $p_1, \ldots, p_k$, if any, are $\mathbb{Q}$-independent). Then the following are equivalent:

- (i) $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ *are totally jointly ergodic for* $(X, \mathcal{B}, \mu, T)$.
- (ii) *There exists* $W_0 \equiv W_0(p_1, \ldots, p_k) \in \mathbb{N}$ *such that* $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ *are* $W!$*-jointly ergodic for* $(X, \mathcal{B}, \mu, T)$ *for all* $W \geqslant W_0$.
- (iii) *There exists an infinite set* $I \equiv I(p_1, \ldots, p_k) \subseteq \mathbb{N}$ *such that* $([p_1(n)])_n, \ldots, ([p_k(n)])_n$ *are* $W!$*-jointly ergodic for* $(X, \mathcal{B}, \mu, T)$ *for all* $W \in I$.
- (iv) $p_1, \ldots, p_k$ *are* $S(T)$*-independent (resp.* $S(T) \setminus \mathbb{Z}_*$*-independent).*

This result gives that Conjecture 2 holds for $k = 1$ in general.

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$([p_1(n)])_n, ([p_2(n)])_n$ are totally jointly ergodic for all totally ergodic systems, but $\big([p_1(Wn+r)]\big)_n, \big([p_2(Wn+r)+1/4]\big)_n$ are not jointly ergodic for $(\mathbb{T}, \mathcal{B}, m, T)$, where m is the Haar measure and $Tx = x + c$ mod 1.

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- Let $c \in \mathbb{R}\backslash\mathbb{Q}$, $W \geqslant 2$, and

$$p_1(n) = n^3 + \frac{cn^2}{4}, \;\; p_2(n) = n^3 + \frac{(c+1)n^2}{4}.$$

$([p_1(n)])_n$, $([p_2(n)])_n$ are $W!$-jointly ergodic for all totally ergodic systems, but not jointly ergodic for $(\mathbb{T}, \mathcal{B}, m, T)$.

Characterizing total joint ergodicity for 2 terms

Theorem (K.-Sun, 2023)

For all $p_1, p_2 \in \mathbb{R}[x]$ with $p_1(0) = p_2(0) = 0$, $([p_1(n)])_n, ([p_2(n)])_n$ are totally jointly ergodic for all totally ergodic systems iff, the following holds:

(i) p_1, p_2 *are $\mathbb{R}\backslash\mathbb{Q}_*$-independent; or*

(ii) $p_1 = f + cg,\ p_2 = \pm(f + (c+1)g)$ *for some $f, g \in \mathbb{Q}[x]$, $f(0) = g(0) = 0$, f is not a multiple of g,[a] g is an integer polynomial with $g \not\equiv 0$, and $c \in \mathbb{R}\backslash\mathbb{Q}$.*

[a] By this we mean that there is no $s \in \mathbb{Q}$ such that $f = sg$.

Approach and why we stop at $k = 2$ terms

Let $(X, \mathcal{B}, \mu, T)$ be a totally ergodic system.

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Let p_1, p_2 are $S(T) \setminus \mathbb{Z}_*$-dependent with $p_1(0) = p_2(0) = 0$.

- If $p_1 \in \mathbb{Q}[x]$ and $p_2 \notin \mathbb{Q}[x]$ (in which case p_2 is $\mathbb{Q}$-independent; the case $p_2 \in \mathbb{Q}[x]$ and $p_1 \notin \mathbb{Q}[x]$ is analogous), or

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then, there exists $W_0 \equiv W_0(p_1, p_2) \in \mathbb{N}$ such that $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$-jointly ergodic for $(X, \mathcal{B}, \mu, T)$ for all $W \geqslant W_0$.

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$$c_1 p_1 + c_2 p_2 = q_1, \;\; d_1 p_1 + d_2 p_2 = q_2$$

for some $q_1, q_2 \in \mathbb{Q}[x]$, $c_1, c_2 \in S(T)\backslash\mathbb{Z}_*$ not both 0, and $d_1, d_2 \in \mathbb{Q}_*$.

More classes of suitable families of functions: Hardy field functions

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  a stays logarithmically away from rational polynomials if $\log x \prec a(x) - p(x)$ for all $p \in \mathbb{Q}[x] + \mathbb{R}$.

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  (These are exactly the Hardy field functions a for which $(a(n))_n$ is equidistributed on $\mathbb{T}$, Boshernitzan, 1994.)

Combinations of Hardy and polynomial functions

Theorem (Frantzikinakis, 2021)

For $k \in \mathbb{N}$, let $a_1, \ldots, a_k : (x_0, +\infty) \to \mathbb{R}$ be Hardy field functions. Suppose that the a_i's and their differences are in $\mathcal{T} + \mathcal{P}$ and every non-trivial linear combination of them with at least one irrational coefficient, stays logarithmically away from rational polynomials. Then, the sequences $([a_1(n)])_n, \ldots, ([a_k(n)])_n$ are (totally) jointly ergodic for every totally ergodic system.

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Thank you for your attention!