# Multiple ergodic averages along polynomials for systems of commuting transformations

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# Multiple ergodic averages with polynomial iterates

We will study $L^2$ limits of averages

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k$$

where

1. $(X, \mathcal{X}, \mu)$ is a standard probability space;
2. $T_1, \ldots, T_k$ are **commuting** invertible measure-preserving transformations acting on $X$
   (I will refer to $(X, \mathcal{X}, \mu, T_1, \ldots, T_k)$ simply as **system**);
3. $p_1, \ldots, p_k \in \mathbb{Z}[n]$ are polynomials with $p_i(0) = 0$ (we call such polynomials **integral**);
4. $f_1, \ldots, f_k \in L^\infty(\mu)$.

# Polynomial Szemerédi theorem (a special case)

**Theorem (Bergelson & Leibman 1996)**

*Let $p_1, \ldots, p_k$ be integral polynomials and $(X, \mathcal{X}, \mu, T_1, \ldots, T_k)$ be a system. Suppose that $\mu(A) > 0$. Then*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T_1^{-p_1(n)}A \cap \cdots \cap T_k^{-p_k(n)}A) > 0.$$

**Corollary**

*Let $p_1, \ldots, p_k$ be integral polynomials and $v_1, \ldots, v_k \in \mathbb{Z}^d$. Then each dense subset of $\mathbb{Z}^d$ contains a polynomial progression of the form*

$$x,\ x + v_1 p_1(n),\ \ldots,\ x + v_k p_k(n)$$

*with $n \neq 0$.*

For instance, each dense subset on $\mathbb{Z}^2$ contains

$$(x_1,\ x_2),\ (x_1 + n,\ x_2),\ (x_1,\ x_2 + n^2)$$

for some $n \neq 0$.

# Norm convergence

**Theorem (Walsh 2012)**

*Let $(X, \mathcal{X}, \mu, T_1, \ldots, T_k)$ be a system, $p_1, \ldots, p_k$ be integral polynomials and $f_1, \ldots, f_k \in L^\infty(\mu)$. Then*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k$$

*converges in $L^2$.*

What can we say about the limit?

# A representative question: joint ergodicity

For instance, when are the polynomials *jointly ergodic* for the system, i.e.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)}f_1\cdots T_k^{p_k(n)}f_k = \int f_1\cdots\int f_k$$

for all functions $f_1, \ldots, f_k$?

# Two ways to describe the limit

We want to find

1. a **factor** ($T$-invariant $\sigma$-algebra) $\mathcal{Y}_j$;
2. or a **seminorm** $\|\cdot\|_j$ such that

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k \to 0$$

if $\mathbb{E}(f_j|\mathcal{Y}_j) = 0$ or $\|f_j\|_j = 0$.

If this happens, we say that $\mathcal{Y}_j$ or $\|\cdot\|_j$ **controls/is characteristic for** the average (at the index $j$).

The factor approach dates back to Furstenberg's multiple recurrence result.

# Host-Kra seminorms

There exist a family of seminorms $\|\!|\cdot|\!\|_{s,T}$ on $L^\infty(\mu)$ satisfying

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} T^n f_1\cdots T^{\ell n} f_\ell\right\|_{L^2(\mu)} \leq C_\ell \min_{j=1,\ldots,\ell} \|\!|f_j|\!\|_{\ell,T}$$

for all 1-bounded functions $f_1,\ldots,f_\ell$.

Similarly, for pairwise distinct integral polynomials $p_1,\ldots,p_\ell$, there exists $s$ such that

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)} f_1\cdots T^{p_\ell(n)} f_\ell\right\|_{L^2(\mu)} = 0$$

whenever $\|\!|f_j|\!\|_{s,T} = 0$ for some $s\in\mathbb{N}$.

These **Host-Kra seminorms** satisfy the monotonicity property

$$\|\!|f|\!\|_{1,T} \leq \|\!|f|\!\|_{2,T} \leq \|\!|f|\!\|_{3,T} \leq \ldots .$$

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# Host-Kra factors

There exist factors $\mathcal{Z}_s(T)$ such that

$$\|f\|_{s,T} = 0 \quad \iff \quad \mathbb{E}(f|\mathcal{Z}_{s-1}(T)) = 0.$$

Properties of the factors:

- The factor $\mathcal{Z}_0(T)$ is the **invariant factor** $\mathcal{I}(T)$;
- In particular, if $T$ ergodic, then $\mathbb{E}(f|\mathcal{Z}_0(T)) = \int f$;
- If $T$ is ergodic, then $\mathcal{Z}_1(T)$ is the **Kronecker factor**;
- For $s \in \mathbb{N}$ and $T$ ergodic, $\mathcal{Z}_s(T)$ is an inverse limit of $s$-step nilsystems (**Host-Kra structure theorem**).

We also need the **rational Kronecker factor**:

$$\begin{aligned}\mathcal{K}_{\mathrm{rat}}(T) &= \langle A \in \mathcal{X} : T^{-r}A = A \text{ for some } r > 0\rangle\\ &= \mathcal{I}(T) \vee \mathcal{I}(T^2) \vee \mathcal{I}(T^3) \vee \cdots .\end{aligned}$$

For instance,

1. For $Tx = x + \sqrt{2}$ on $\mathbb{T}$, we have

$$\mathbb{E}(f|\mathcal{K}_{\mathrm{rat}}(T)) = \mathbb{E}(f|\mathcal{I}(T)) = \int f;$$

2. For $Tx = x + 1$ on $\mathbb{Z}/N\mathbb{Z}$, we have

$$\mathbb{E}(f|\mathcal{K}_{\mathrm{rat}}(T)) = f \quad \text{whereas} \quad \mathbb{E}(f|\mathcal{I}(T)) = \int f.$$

Hence $\mathcal{I}(T) = \mathcal{Z}_0(T) \subset \mathcal{K}_{\mathrm{rat}}(T) \subset \mathcal{Z}_1(T) \subset \mathcal{Z}_2(T) \subset \cdots$

# Single averages

If $T$ is **totally ergodic** (i.e. $T, T^2, \ldots$ are ergodic), then

$$\frac{1}{N}\sum_{n=1}^{N} T^{p(n)}f \to \int f;$$

For general $T$, we have

$$\frac{1}{N}\sum_{n=1}^{N} T^{p(n)}f \to 0 \quad \text{if} \quad \mathbb{E}(f|\mathcal{K}_{\mathrm{rat}}(T)) = 0.$$

On the combinatorial side, this corresponds to the fact that if $f_0, f_1 : \mathbb{Z} \to \mathbb{C}$ are 1-bounded and compactly supported and

$$\sum_{x\in\mathbb{Z}} \frac{1}{N}\sum_{n=1}^{N} f_0(x)f_1(x+p(n))$$

is “large”, then $f_1$ has a large average on **arithmetic progressions** of small common difference and length $\sim N^{\deg p}$.

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# Multiple ergodic averages of a single transformation

For distinct $p_1, \ldots, p_k$, we know a lot about the $L^2$ limits of

$$\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k :$$

1. They are always controlled by some Host-Kra seminorm (Host & Kra 2005);
2. If $p_1, \ldots, p_k$ are **linearly independent**, then $\mathcal{K}_{\mathrm{rat}}(T)$ is characteristic (Frantzikinakis & Kra 2006);
3. If $p_1, \ldots, p_k$ are **"quadratically independent"**, then $\mathcal{Z}_1(T)$ is characteristic (K. 2022);
4. When $T$ is **weakly mixing**, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k = \int f_1 \cdots \int f_k$$

for all distinct polynomials (Bergelson 1987).

# Summary of our results

For commuting $T_1, \ldots, T_k$ and integral $p_1, \ldots, p_k$, we examine the $L^2$ limits of

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k,$$

proving:

1. Limiting formulas when $p_1, \ldots, p_k$ are linearly independent or $T$ is weak mixing;
2. Optimal multiple recurrence results;
3. Control by Host-Kra seminorms in most cases of interest;
4. Nil + null decomposition for multicorrelation sequence.

# Linearly independent polynomials

**Theorem (Frantzikinakis & K. 2022)**

*Let $p_1, \ldots, p_k$ be* **linearly independent** *integral polynomials.*

1. *If $T_1, \ldots, T_k$ are* **totally ergodic**, *then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k = \int f_1 \cdots \int f_k.$$

2. *For general commuting $T_1, \ldots, T_k$, the factors $\mathcal{K}_{\mathrm{rat}}(T_j)$ are characteristic.*

Part (1) only known before for **distinct degree** polynomials (due to Chu, Frantzikinakis & Host 2011).

Not even previously known for

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T_1^{n^2} f_1 \cdot T_2^{n^2+n} f_2.$$

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# Optimal multiple recurrence

**Corollary (Frantzikinakis & K. 2022)**

1. *If* $p_1, \ldots, p_k$ *are* **linearly independent** *integral polynomials, then for every* $A \subset X$ *and* $\varepsilon > 0$*, we have*

$$\mu(A \cap T_1^{-p_1(n)} A \cap \cdots \cap T_k^{-p_k(n)} A) \geq \mu(A)^{k+1} - \varepsilon$$

   *for a syndetic set of n.*

2. *Likewise, for every* $B \subset \mathbb{Z}^d$*, directions* $v_1, \ldots, v_k \in \mathbb{Z}^d$ *and* $\varepsilon > 0$*, we have*

$$\overline{d}(B \cap (B - v_1 p_1(n)) \cap \cdots \cap (B - v_k p_k(n))) \geq \overline{d}(B)^{k+1} - \varepsilon$$

   *for a syndetic set of n.*

# Proof components

Our proof uses two main properties of linearly independent polynomials:

1. **Equidistibution property**: for every $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ not all rational and any linearly indepdendent integral polynomials $p_1, \ldots, p_k$,

$$\alpha_1 p_1(n) + \cdots + \alpha_k p_k(n)$$

   is equidistributed in $\mathbb{T}$; in particular, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(\alpha_1 p_1(n) + \cdots + \alpha_k p_k(n)) = 0.$$

2. **Seminorm estimates** for the limits

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k.$$

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# Seminorm estimates for commuting transformations

**Theorem (Frantzikinakis & K. 2022)**

*Let $p_1, \ldots, p_k$ be* **pairwise linearly independent**. *There exists $s \in \mathbb{N}$ such that for every system, we have*

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k \right\|_{L^2(\mu)} = 0$$

*whenever $\| f_i \|_{s, T_i} = 0$ for some $i$.*

Previously, this was only known for distinct degree polynomials.

It was not even known for

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T_1^{n^2} f_1 \cdot T_2^{n^2+n} f_2.$$

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# Corollary for weakly mixing transformations

**Corollary (Frantzikinakis & K. 2022)**

*Let $p_1, \ldots, p_k$ be* **pairwise linearly independent**. *For every system of* **weakly mixing** *transformations, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_k^{p_k(n)} f_k = \int f_1 \cdots \int f_k,$$

This is because for a weakly mixing transformation $T$, we have

$$|||f|||_{s,T} = \left|\int f\right|$$

for any $s \in \mathbb{N}$ and bounded $f$.

# Seminorm control for pairwise dependent polynomials

What about an average like

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T_1^{n^2}f_1\cdot T_2^{n^2}f_2\cdot T_3^{n^2+n}f_3?$$

We get seminorm control whenever

$$\mathcal{I}(T_1^{-1}T_2)=\mathcal{I}(T_1)\cap\mathcal{I}(T_2)$$

(so in particular when $T_1^{-1}T_2$ is ergodic).

We show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{n^2} f \cdot S^{n^2+n} g = \int f \cdot \int g$$

whenever $T, S$ are totally ergodic and commute.

1. Box seminorm control;
2. Host-Kra seminorm control;
3. Degree lowering.

# Box seminorms

For $T, S$, we define

$$\|\!|f|\!\|_{T,S}^4 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \int f \cdot T^n\overline{f} \cdot S^m\overline{f} \cdot T^nS^m f.$$

We can similarly define $\|\!|f|\!\|_{T_1,\ldots,T_s}$.

# From box seminorms to Host-Kra seminorms

Donoso, Ferré-Moragues, Koutsogiannis and Sun showed that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{n^2} f \cdot S^{n^2+n} g = 0 \qquad (1)$$

whenever $\|\|g\|\|_{S,\ldots,S,T^{-1}S,\ldots,T^{-1}S} = 0$.

Our input: (1) holds whenever $\|\|g\|\|_{s,S} = 0$ for some $s \in \mathbb{N}$ via a new **seminorm smoothing** technique.

This stronger seminorm control is necessary for the degree lowering argument.

# Seminorm smoothing

A **ping-pong** strategy: we pass information from $g$ to $f$ and then back to $g$.

Suppose for simplicity that we have $\|g\|_{T^{-1}S}$ control, i.e.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^{n^2}f\cdot S^{n^2+n}g=0\quad\text{whenever}\quad \|g\|_{T^{-1}S}=0.$$

1. (**Ping**) Control by $\|f\|_{T,\ldots,T}$;
2. (**Pong**) Control by $\|g\|_{S,\ldots,S}$.

# Ping

Suppose that

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}T^{n^2}f\cdot S^{n^2+n}g\right\|_2>0.$$

Let $h=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^{n^2}f\cdot S^{n^2+n}g$ and

$$G=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}S^{-(n^2+n)}\overline{h}\cdot S^{-(n^2+n)}T^{n^2}\overline{f},$$

so that

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}T^{n^2}f\cdot S^{n^2+n}g\right\|_2^2=\int g\cdot G.$$

By the Cauchy-Schwarz and definition of $G$, we have

$$\|G\|_2^2=\lim_{N\to\infty}\int h\cdot\frac{1}{N}\sum_{n=1}^{N}T^{n^2}f\cdot S^{n^2+n}G>0.$$

and so

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}T^{n^2}f\cdot S^{n^2+n}G\right\|_2>0.$$

Recall the assumption that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{n^2} f \cdot S^{n^2+n} g = 0 \quad \text{whenever} \quad \|g\|_{T^{-1}S} = 0$$

for **all** $g$. In particular,

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{n^2} f \cdot S^{n^2+n} g \right\|_2 > 0$$

$$\Rightarrow \lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{n^2} f \cdot S^{n^2+n} G \right\|_2 > 0 \Rightarrow \|G\|_{T^{-1}S} > 0.$$

Importantly, we have

$$\|G\|^2_{T^{-1}S} = \int G \cdot u > 0$$

for a $T^{-1}S$-invariant function $u$ (i.e. $Su = Tu$); hence

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{n^2} f \cdot S^{n^2+n} u \right\|_2 = \lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{n^2} f \cdot T^{n^2+n} u \right\|_2 > 0.$$

This we know how to handle. We get $\|f\|_{s,T} > 0$ for some $s \in \mathbb{N}$.

So in the **ping** step, we started with averages of the form

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{n^2}f \cdot S^{n^2+n}g$$

and ended up with averages

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{n^2}f \cdot T^{n^2+n}u$$

that are simpler to handle.

In the **pong** step, we similarly reduce to averages

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{n^2}(\mathcal{D}_{s,T}f) \cdot S^{n^2+n}g$$

that we know how to handle ($\mathcal{D}_{s,T}f$ is the **dual function** associated with $\|\!|f|\!\|_{s,T}$).

# Longer averages

For longer averages such as

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{n^2}f \cdot S^{n^2+n}g \cdot R^{n^2+2n}h,$$

we induct on complexity.

In the **ping** step, we reduce to $T, S, S$, then $S, S, S$ (same transformation = base case).

In the **pong** step, we replace functions by dual functions (these can be removed by van der Corput).

# Adaptations to combinatorics

**Theorem (K. 2023)**

*Let $p_1, \ldots, p_k$ be* **linearly independent** *integral polynomials and $v_1, \ldots, v_k \in \mathbb{Z}^d$. Then each subset of $(\mathbb{Z}/N\mathbb{Z})^d$ ($N$ prime) lacking a polynomial progression of the form*

$$m, \; m + v_1 p_1(n), \; \ldots, \; m + v_k p_k(n)$$

*with $n \neq 0$ has at most $O(N^{d-c})$ elements.*

This jointly generalises results of Peluse (for $d = 1$) and myself (for distinct degree polynomials).

This extension uses a **quantitative concatenation** result for box norms of independent interest.