A non-flag arithmetic regularity lemma and counting lemma

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Notation and the arithmetic regularity lemma

- Let $T_{4AP}(f) := \mathbb{E}_{x,d} f(x)f(x+d)f(x+2d)f(x+3d)$, where
  - $[N] := \{1, 2, \ldots, N\}$,
  - $\mathbb{E}_{x,d} := \frac{1}{N^2} \sum_{x,d \in [N]}$,
  - $f : \mathbb{N} \to \mathbb{C}$ are 1-bounded.

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- In general, let $\Psi = (\psi_1, \ldots, \psi_t)$, each a linear form mapping $\mathbb{Z}^D \to \mathbb{Z}$, define $T_\Psi(f) := \mathbb{E}_{x_1,\ldots,x_D} \prod_{i=1}^{t} f(\psi_i(x_1, \ldots, x_D))$.
  (e.g. $t = 4$, $D = 2$,
  $\psi_1(x, d) = x, \psi_2(x, d) = x+d, \psi_3(x, d) = x+2d, \psi_4(x, d) = x+3d$)

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Theorem (Arithmetic regularity lemma, very informally)

Let $f : [N] \to \mathbb{C}$ be 1-bounded. Then there exists a finite complexity nilsequence f_{nil} such that

$$T_\Psi(f) \approx T_\Psi(f_{\text{nil}}).$$

New goal: understand $T_\Psi(f_{\text{nil}})$.

What is a nilsequence?

Additive/linear characters

$e(\theta n)$, where:

1. θn is a 'linear sequence',
2. on the simply-connected abelian Lie group $\mathbb{R}$,
3. which has cocompact lattice $\mathbb{Z}$,
4. and $e(\cdot)$ is a smooth function on $\mathbb{R}$ which is automorphic with respect to $\mathbb{Z}$.

Polynomial nilsequences

$F(g(n)\Gamma)$, where:

1. $g(n) = g_1^n g_2^{n^2} \cdots g_s^{n^s}$ is a polynomial sequence,
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E.g.: $e(\alpha n + \beta n^2 + \theta n^{100})$

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E.g.: $G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$, $g(n) = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & \alpha n & \alpha\beta\binom{n}{2} \\ 0 & 1 & \beta n \\ 0 & 0 & 1 \end{pmatrix}$

...yields... $e(\alpha n \lfloor \beta n \rfloor)$

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$$\mathbb{E}_{n\in[N]}F(\frac{1}{3}n) \approx \frac{1}{3}\left(F(0)+F(1/3)+F(2/3)\right).$$

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- What about $(\sqrt{2}n, (\sqrt{2}+\frac{1}{3})n)$ in $\mathbb{R}^2/\mathbb{Z}^2$?

Equidistribution on cosets of a subnilmanifold

G simply-connected, nilpotent Lie group, Γ cocompact lattice, $g(n)$ polynomial sequence in G.

Theorem (Leibman, Green–Tao)

The sequence $g(n)\Gamma$ equidistributes in G/Γ if and only if, for all nontrivial, continuous homomorphisms $\eta : G \to \mathbb{R}$ which map Γ to $\mathbb{Z}$, $\eta(g(n)\Gamma)$ is not constant.

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Corollary

We may factorise $g = \varepsilon g' \gamma$ where $\varepsilon \in G$ is a constant, g' is a polynomial sequence which equidistributes in some subnilmanifold $G'/(G' \cap \Gamma)$ of G, and $\gamma(n)\Gamma$ is periodic.

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(Think: $(\sqrt{2}n, (\sqrt{2} + \frac{1}{3})n) = (\sqrt{2}, \sqrt{2})n + (0, \frac{1}{3})n$.)

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(Think: $(\sqrt{2}n, (\sqrt{2} + \frac{1}{3})n) = (\sqrt{2}, \sqrt{2})n + (0, \frac{1}{3})n$.)
Understand $\mathbb{E}_{n \leq N} F(g(n)\Gamma)$, by writing $\mathbb{E}_{n \leq N} F(g'(n)\gamma(n)\Gamma)$, foliating into subprogressions on which $\gamma(n)\Gamma$ is constant, and then analysing $\int_{G'\gamma_i/\Gamma} F d\mu$.
(Think: $(\sqrt{2}n, (\sqrt{2} + \frac{1}{3})n)$, on (x, x), $(x, x) + (0, 1/3)$, $(x, x) + (0, 2/3)$.)

Distribution on linear patterns

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- View $f_{\mathrm{nil}}(\psi_1(x_1,\ldots,x_D)) \cdots f_{\mathrm{nil}}(\psi_t(x_1,\ldots,x_D))$ as a multiparameter nilsequence on G^t/Γ^t with polynomial sequence $g^\Psi(x_1,\ldots,x_D) = (g(\psi_1(x_1,\ldots,x_D)),\ldots,g(\psi_t(x_1,\ldots,x_D))$ and automorphic function $F^{\otimes t}$.

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- By multiparameter versions of the Leibman/Green–Tao equidistribution theorems g^Ψ equidistributes in cosets of a subnilmanifold of G^t/Γ^t.

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- By multiparameter versions of the Leibman/Green–Tao equidistribution theorems g^Ψ equidistributes in cosets of a subnilmanifold of G^t/Γ^t. Then: $T_\Psi(f) \approx T_\Psi(f_{\mathrm{nil}}) \approx \mathbb{E}_{\mathrm{subprogs}} \int_{G^{(g,\Psi)}{}_{\gamma_i}/\Gamma^t} F^{\otimes t}$.

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- New goal: describe the subprogressions and $G^{(g,\Psi)}/\Gamma^t$ given g,Ψ. What properties does $G^{(g,\Psi)}/\Gamma^t$ have?

Result in the flag case

Where does $g^{\Psi}(\boldsymbol{x}) := (g(\psi_1(\boldsymbol{x})), \cdots, g(\psi_t(\boldsymbol{x}))) = g_1^{\Psi(x)} g_2^{\Psi(x)^2} \cdots g_s^{\Psi(x)^s}$ distribute in G^t/Γ^t?

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- Definition: Let V^i be the smallest vector space containing $\Psi(\boldsymbol{x})^i$.
- $\Psi(x, d) = (x, x+d, x+2d, x+3d) = (1,1,1,1)x + (0,1,2,3)d$. Then $V = \text{span}((1,1,1,1),(0,1,2,3))$, and $V^2 = \text{span}((1,1,1,1),(0,1,2,3),(0,1,4,9))$. Also $V^3 = \mathbb{R}^4$.

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Theorem (Green–Tao)

Given any g we may factorise $g = g'\gamma$ where g' is irrational with respect to some $G'_\bullet$ (potentially in a subgroup), and γ is rational.

A non-flag example

- Consider $\Psi(x, y) = (y, 2x + 2y, x + 3y, x) = (0, 2, 1, 1)x + (1, 2, 3, 0)y =: v_1 x + v_2 y$. Then $V = \text{span}\{(0, 2, 1, 1), (1, 2, 3, 0)\}$

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- Consider on the Heisenberg group the linear polynomial sequence
$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & an & cn + \binom{n}{2}ab \\ 0 & 1 & bn \\ 0 & 0 & 1 \end{pmatrix}.$$

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- The Leibman group is $G^\Psi = \langle G^V, G_2^{V^2} \rangle \leq G^4$.
- Define η on $G^\Psi \leq G^4$ by $\eta(h_1,h_2,h_3,h_4) = w \cdot (c_1,c_2,c_3,c_4)$, where $$h_i = \begin{pmatrix} 1 & a_i & c_i \\ 0 & 1 & b_i \\ 0 & 0 & 1 \end{pmatrix} \in G \text{ for each } i \text{ and } w = (24,3,-4,-8) \in V^{2\perp}.$$

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- Then $\eta(g^\Psi(x,y)) = w \cdot v_1(c - \frac{1}{2}ab)x + w \cdot v_2(c - \frac{1}{2}ab)y$.

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$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & an & cn + \binom{n}{2}ab \\ 0 & 1 & bn \\ 0 & 0 & 1 \end{pmatrix}.$$
- The Leibman group is $G^\Psi = \langle G^V, G_2^{V^2} \rangle \leq G^4$.
- Define η on $G^\Psi \leq G^4$ by $\eta(h_1,h_2,h_3,h_4) = w \cdot (c_1,c_2,c_3,c_4)$, where $h_i = \begin{pmatrix} 1 & a_i & c_i \\ 0 & 1 & b_i \\ 0 & 0 & 1 \end{pmatrix} \in G$ for each i and $w = (24,3,-4,-8) \in {V^2}^\perp$.
- Then $\eta(g^\Psi(x,y)) = w \cdot v_1(c - \frac{1}{2}ab)x + w \cdot v_2(c - \frac{1}{2}ab)y$.
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A non-flag example

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- What data from a polynomial sequence g determines the distribution of g^Ψ? Spoiler answer: the (additive) distribution of the coefficients of g in the Lie algebra. (modulo factorisation).

Qualitative equidistribution in the Lie algebra

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- Defn: A polynomial sequence p in $\mathfrak{g}$ equidistributes in $(\mathfrak{g}, \mathfrak{g}_{\mathbb{Q}})$ if $\exp p$ equidistributes in G/Γ, where Γ is *any* lattice such that $\operatorname{span}_{\mathbb{Q}} \log \Gamma = \mathfrak{g}_{\mathbb{Q}}$.

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Theorem (Lie algebra multiparameter Leibman/Green–Tao)

$p(\boldsymbol{n})$ in $\mathfrak{g}$ equidistributes in $(\mathfrak{g}, \mathfrak{g}_{\mathbb{Q}})$ if and only if for every nontrivial rational Lie algebra homomorphism $\eta : \mathfrak{g} \to \mathbb{R}$, we have $\eta \circ p(\boldsymbol{n}) \not\subset \mathbb{Q}$.

Qualitative equidistribution on linear patterns

Let $p(n) = \sum_{i=1}^{s} a_i n^i$ be a polynomial sequence in $\mathfrak{g}$. Let Ψ be a linear pattern. Where does $p^{\Psi}(\boldsymbol{x}) = (p(\psi_1(\boldsymbol{x}), \ldots, p(\psi_t(\boldsymbol{x})))$ distribute in $\mathfrak{g}^t$?

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- Definition: $a \in \mathfrak{g}$ is *linearly irrational* wrt $\mathfrak{g}_{\mathbb{Q}}$ if for all rational linear maps $l \in \mathfrak{g}_{\mathbb{Q}}^*$, we have $l(a) \in \mathbb{Q} \implies l(a) = 0$.

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Theorem

Let $p(n) = \sum_{i=1}^{s} a_i n^i$ be a polynomial sequence in $\mathfrak{g}$ with rational structure $\mathfrak{g}_\mathbb{Q}$ such that a_i is linearly irrational for each i. Let Ψ be a linear pattern. Then p^Ψ equidistributes in $\mathfrak{g}^\Psi := \langle S_i \otimes V^i \rangle$, where S_i is the smallest rational subspace containing a_i.

Qualitative equidistribution on linear patterns, cont.

Theorem (A. 2022+)

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Proof:

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Theorem (A. 2022+)

For any polynomial sequence p in G/Γ, there is a factorisation $p = cp'\gamma$, where c is a constant, p' is linearly irrational, and γ is periodic mod Γ.

The end