Free nilspaces, double-coset nilspaces, and Gowers norms

Diego González-Sánchez (joint work with Pablo Candela and Balázs Szegedy)

Alfréd Rényi Institute of Mathematics

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Introduction

The study of Gowers norms (Gowers, 2001) has produced many interesting results in the last years, the most notable of which being the proof that the primes contain arbitrarily long arithmetic progression (Green and Tao, 2008).

A central topic in the area is about the inverse theorems for the Gowers norms. For finite cyclic groups this was achieved in a series of papers (Green and Tao 2008, Green, Tao and Ziegler 2011, 2012).

Similarly, for vector spaces of characteristic p this was done in (Bergelson, Tao and Ziegler 2010, Tao and Ziegler 2010, 2012) with a recent qualitative refinement recently (Berger, Sah, Sawhney and Tidor, 2021).

Recent developments

Theorem (Jamneshan and Tao, 2021, informal)

Let Z be a finite abelian group and $f \in L^\infty(Z)$ a 1-bounded function. Suppose that $\|f\|_{U^3(Z)} \geq \epsilon$ for some fixed $\epsilon > 0$. Then there exists a degree-2 filtered nilmanifold H/Γ, a polynomial map $\varphi : Z \to H/\Gamma$ and a Lipschitz function $W : H/\Gamma \to \mathbb{C}$ such that $|\langle f, W \circ \varphi \rangle| \gg_\epsilon 1$.[a]

[a] This result is even more explicit in terms of of the implicit constant $\gg_\epsilon$ and the complexity of the nilmanifold H/Γ.

Theorem (Jamneshan, Shalom and Tao, 2023, informal)

Let Z be a finite abelian m-torsion group and $f \in L^\infty(Z)$ a 1-bounded function. Suppose that $\|f\|_{U^{k+1}(Z)} \geq \epsilon$ for some fixed $\epsilon > 0$. Then there exists a polynomial map P on Z of degree at most $O(k, m)$ such that $|\langle f, e(-P) \rangle| \gg_\epsilon 1$.

Introduction, cont.

In ergodic theory, seminorms analogous to Gowers norms were introduced (Host and Kra 2005). (Host and Kra 2008) initiated an axiomatic approach for Gowers norms defining *the weakest structure a set must have so that one can define a 2(3)-Gowers norm*. Inspired by this, (Antolín Camarena and Szegedy 2012) introduced the concept of *nilspaces*. Within this framework, a general inverse theorem can be proved for any finite abelian group:

Theorem (Candela and Szegedy, 2019, informal)

Let Z *be a finite abelian group and* $f \in L^\infty(\mathrm{Z})$ *a 1-bounded function. Suppose that* $\|f\|_{U^{k+1}(\mathrm{Z})} \geq \epsilon$ *for some fixed* $\epsilon > 0$*. Then there exists a k-step* CFR *nilspace* X*, a morphism* $\varphi : \mathrm{Z} \to \mathrm{X}$ *and a Lipschitz function* $W : \mathrm{X} \to \mathbb{C}$ *such that* $|\langle f, W \circ \varphi\rangle| \gg_\epsilon 1$*.*

Can we give a description of X, φ and W in terms of more familiar objects?

CFR nilspaces as quotients by higher-order lattice actions

Fourier analysis: Corresponds to the U^2 norm. The correlating objects function $F \circ \varphi$ can be taken to be

$$\chi : Z \xrightarrow{\ \varphi\ } \mathbb{R}/\mathbb{Z} \xrightarrow{\ e(\cdot)\ } \mathbb{C},$$

where φ is a homomorphism and $e(x) := e^{2\pi i x}$, i.e., χ is a Fourier character.

Higher order Fourier analysis: This picture can be extended to

$$\chi : Z \xrightarrow{\ \varphi\ } (\mathbb{Z}^r \times \mathbb{R}^s)/\Gamma \xrightarrow{\ W\ } \mathbb{C},$$

where $(\mathbb{Z}^r \times \mathbb{R}^s)/\Gamma$ is the quotient space of $\mathbb{Z}^r \times \mathbb{R}^s$ by the action of Γ which is a group described in terms of polynomials acting on $\mathbb{Z}^r \times \mathbb{R}^s$ (NOT a subgroup of $\mathbb{Z}^r \times \mathbb{R}^s$ in general). W is a Lipschitz map and φ a morphism.

Example: The Heisenberg manifold

The Heisenberg manifold appears in many works in higher order Fourier analysis (Green and Tao, 2006). It equals $\mathcal{H}/\Gamma$ where $\mathcal{H} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$ is the Heisenberg group and $\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$ is the *discrete* Heisenberg group.

In the above framework we can describe it as follows: On $\mathbb{R}^3$ we define Γ as the group generated by $(x, y, z) \mapsto (x + 1, y, z)$ and $(x, y, z) \mapsto (x, y + 1, z + x)$. Then as a nilspace the Heisenberg manifold equals $\mathbb{R}^3/\Gamma$.

Free nilspaces and their translation groups

Definition (Free nilspace)

A free nilspace is a group nilspace of the form $F = \prod_{i=1}^{k} \mathcal{D}_i(\mathbb{Z}^{a_i} \times \mathbb{R}^{b_i})$ where $\mathcal{D}_i(\mathrm{Z})$ is the group nilspace generated by the filtration $\mathrm{Z}_\bullet = (\mathrm{Z}_j)_{j \geq 0}$, $\mathrm{Z}_j = \mathrm{Z}$ if $j \leq i$ and $\{0\}$ otherwise with the Host-Kra cubes.

Definition (Translation group of a free nilspace)

Let F be a free nilspace. The group $\Theta(F)$ is the group of translations that preserve cubes when applied on faces of a certain codimension.

Γ is a subgroup of $\Theta(F)$, the translation group of a free nilspace F.

Example: If $F = \mathcal{D}_1(\mathbb{Z}) \times \mathcal{D}_2(\mathbb{R}) \times \mathcal{D}_3(\mathbb{Z})$ we have that $\Theta(F)$ is the group of transformations $(x, y, z) \to (x + a, y + b + cx, z + d + ex + f\binom{x}{2})$ for $a, d, e, f \in \mathbb{Z}$ and $b, c \in \mathbb{R}$.

Fiber-transitive group actions

On a free nilspace $F = \prod_{i=1}^{k} \mathcal{D}_i(\mathbb{Z}^{a_i} \times \mathbb{R}^{b_i})$ we define $\pi_j : F \to \prod_{i=1}^{j} \mathcal{D}_i(\mathbb{Z}^{a_i} \times \mathbb{R}^{b_i})$ as the map that *forgets* the components of degree larger than j.

Definition (Fiber-transitive group of translations)

Let F be a k-step free nilspace and let $\Gamma \leq \Theta(F)$. We say that Γ is a *fiber-transitive group on F* if the following holds: for all $x, y \in F$, if there exists $\gamma \in \Gamma$ such that $\gamma(x) = y$ and $\pi_j(x) = \pi_j(y)$ for some $j \in [k]$, then there exists $\gamma' \in \Gamma \cap \Theta_{j+1}(F)$ such that $\gamma'(x) = y$.

Example: (Host and Kra, 2008) On $F = \mathcal{D}_1(\mathbb{Z}) \times \mathcal{D}_2(\mathbb{Z})$ let $\Gamma = \langle (x, y) \mapsto (x + p, y + x), (x, y) \mapsto (x, y + p), (x, y) \mapsto (x, y + px) \rangle$.

Non-example: On the same F as before, $\Gamma' := \langle (x, y) \mapsto (x, y + x) \rangle$.

CFR nilspaces as quotients of a free nilspace by a fiber-transitive group action

Observation: For 1-step free nilspaces $F = \mathcal{D}_1(\mathbb{Z}^a \times \mathbb{R}^b)$, a fiber-transitive $\Gamma \leq \Theta(F)$ is just a subgroup of $\mathbb{Z}^a \times \mathbb{R}^b$ (as an abelian group). In fact, it is true that any compact *finite-rank* abelian group is the quotient of $\mathbb{Z}^a \times \mathbb{R}^b$ by a cocompact lattice.

Theorem (Candela, G-S, Szegedy)

Let X *be a* k*-step compact finite-rank nilspace. Then there exists a* k*-step free nilspace* F*, and a fiber-transitive group* $\Gamma \subset \Theta(F)$*, such that* $\mathrm{X} \cong F/\Gamma$*.*

Corollary (Inverse theorem, informal)

Combining this with the inverse theorem of Candela and Szegedy (2019) we prove that usual Fourier characters are replaced by functions of the form $\chi : \mathrm{Z} \xrightarrow{\varphi} F/\Gamma \xrightarrow{W} \mathbb{C}$ *for the* U^{k+1} *norm.*

Double coset nilspaces

The space F/Γ is a quotient but Γ is not (in general) a subgroup of F. However, there is a natural description of this quotient in terms of *double coset spaces*. Representing nilspaces as double cosets was suggested by Gutman, Manners and Varjú (private communication 2014) and in ergodic theory double cosets have been successfully used to describe the 2-factors of some ergodic systems (Shalom 2021).

Let $x_0 \in F$ be arbitrary and let $\mathrm{Stab}(x_0) := \{\alpha \in \Theta(F) : \alpha(x_0) = x_0\}$. It can be proved that (as a nilspace) $F \cong \mathrm{Stab}(x_0) \backslash \Theta(F)$. Thus:

Theorem (Candela, G-S, Szegedy (informal))

Let F be a free nilspace and Γ a fiber-transitive group action. Let $x_0 \in F$ be any fixed point. Then $F/\Gamma \cong \mathrm{Stab}(x_0) \backslash \Theta(F)/\Gamma$ as a double coset space.

Double coset nilspaces, cont.

The fiber-transitive property for the group Γ translates to a nice and symmetric property of a pair of subgroups $K, \Gamma \leq G$ in order for $K \backslash G / \Gamma$ to be a nilspace:

Definition (Groupable nilpair)

Let $(G, G_\bullet)$ be a filtered group of degree k and let K, Γ be subgroups of G. We say that (K, Γ) is a *groupable nilpair* in $(G, G_\bullet)$ if any of the following equivalent properties is satisfied:

1. For every $x \in G$ and every $i \geq 0$ we have $(Kx\Gamma) \cap (G_i x\Gamma) = (K \cap G_i)x\Gamma$.
2. For every $x \in G$ and every $i \geq 0$ we have $(Kx\Gamma) \cap (KxG_i) = Kx(\Gamma \cap G_i)$.

Remark: In parallel with the completion of this work, Jamneshan, Shalom and Tao (2023) shared with us a preprint where they prove that totally disconnected Γ-systems of order k are represented as double coset spaces satisfying the previous definition.

Let us discuss about the topology of all these objects...

Even for topological abelian groups problems may arise when defining quotients:

Example: In the abelian group $\mathbb{R}$ consider the subgroups $\mathbb{Z}$ and $\mathbb{Q}$. Both are normal subgroups so algebraically both quotients are well-defined abelian groups. However, $\mathbb{R}/\mathbb{Z}$ is much nicer than $\mathbb{R}/\mathbb{Q}$ (with the quotient topology).

From the perspective of nilspaces we face similar problems. In fact, it was not clear at all even how to define a topological nilspace with a non-compact topology.

LCH nilspaces

We will say that a topological space is LCH if it is locally-compact, second-countable and Hausdorff.

Definition (LCH nilspace)

We say that a nilspace X is an LCH *nilspace* if X is an LCH topological space such that the following property holds for every integer $n \geq 0$:

1. The cube set $C^n(X)$ is closed in the product topology on $X^{[\![n]\!]}$.
2. The coordinate projection $p^{[\![n]\!]} : C^n(X) \to \mathrm{Cor}^n(X)$ is an open map.

Sanity check: If we did not have condition 2, we could define a 1-step LCH nilspace which is not a topological abelian group.

More on topology

Once this definition is established, many objects used in the theory of nilspaces have to be studied from the perspective of having a topology. To name a few:

1. Extension of nilspaces: if $p : \mathrm{Y} \to \mathrm{X}$ is an (algebraic) extension of nilspaces, what conditions on X and p ensure that Y is also an LCH nilspace?
2. The group of translations $\Theta(\mathrm{X})$ of an LCH nilspace is now defined as the group of *continuous* translations. This forces the group of translations of a free nilspace F to be describable in terms of polynomials as stated before.
3. For fiber-transitive group actions we need to find conditions analogous to those for topological groups to ensure well-defined LCH quotients.

Slogan: Topology helps us in many parts of the proof albeit requiring us to extend many existing results to topological setting.

Extensions of nilspaces

Definition (Continuous extensions of LCH nilspaces)

Let $k, t \geq 0$ be integers. Let X, Y be k-step LCH nilspaces such that Y is an algebraic nilspace extension of X of degree t, by some LCH abelian group Z, with associated projection $p : \mathrm{Y} \to \mathrm{X}$. We say that Y (or $p : \mathrm{Y} \to \mathrm{X}$) is a *continuous extension* of X if the action of Z on Y is continuous and p is a continuous open map.

Observation: This is analogous to the theory of extensions of abelian LCH groups. In fact, any surjective homomorphism $\varphi : G \to H$ defines the short exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow G \longrightarrow H \longrightarrow 0.$$

Extensions of free nilspaces

Recall: Let G be an abelian Lie group and $\varphi : G \to \mathbb{Z}^r \times \mathbb{R}^s$ a surjective (continuous) homomorphism. Then $G \cong \mathbb{Z}^r \times \mathbb{R}^s \times \ker(\varphi)$ (that is, the group $\mathbb{Z}^r \times \mathbb{R}^s$ is projective for Lie groups).

Theorem (Candela, G-S, Szegedy)

Let F be a k-step free nilspace and let Y be a degree-k extension of F by an abelian Lie group Z, with corresponding projection $q : \mathrm{Y} \to F$. Then this extension splits, i.e., there exists a continuous morphism $s : F \to \mathrm{Y}$ such that $q \circ s = \mathrm{id}$. In particular Y is isomorphic as an LCH *nilspace to the product-nilspace $F \times \mathcal{D}_k(\mathrm{Z})$.*

Corollary: the Heisenberg group $\mathcal{H} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$ endowed with the Host-Kra cubes is isomorphic as an LCH nilspace to $\mathcal{D}_1(\mathbb{R}^2) \times \mathcal{D}_2(\mathbb{R})$.

Extensions of free nilspaces, cont.

Proof sketch: Induction on the step k of $F = \prod_{i=1}^{k} \mathcal{D}_i(\mathbb{Z}^{a_i} \times \mathbb{R}^{b_i})$.

1. Prove that the map q is *consistent*, i.e., for every $i \in [k]$ and $\alpha \in \Theta_i(\mathrm{Y})$ there exists (a unique) $\beta \in \Theta_i(F)$ such that $q \circ \alpha = \beta \circ q$. Let $\widehat{q} : \Theta(\mathrm{Y}) \to \Theta(F)$ be the map sending $\alpha \mapsto \beta$.
2. Prove that the map $\widehat{q}$ is surjective. To do so we use the criteria of *Translation bundles* (Antolín-Camarena and Szegedy, 2012) which reduces the problem to check if some *other* extension $\mathcal{T}_{k-1}$ of $\pi_{k-1}(F) = F_{k-1}$ splits. This is given by the induction hypothesis.
3. For $i \in [k]$ and $j \in [a_i]$, let $\beta_{i,j} \in \Theta_i(F)$ the map that adds 1 in the jth coordinate of $\mathcal{D}_i(\mathbb{Z}^{a_i})$ and let $\beta'_{i,j} \in \Theta_i(\mathrm{Y})$ be such that $\widehat{q}(\beta'_{i,j}) = \beta_{i,j}$. Similarly, let $\gamma_{i,\ell}(c) : \mathbb{R} \to \Theta_i(F)$ the 1-parameter subgroup that adds c on the ℓth coordinate of $\mathcal{D}_i(\mathbb{R}^{b_i})$ and let $\gamma'_{i,\ell}(c) : \mathbb{R} \to \Theta_i(\mathrm{Y})$ be a lift of it.
4. Any $x \in F$ equals uniquely $\prod_{i,j,\ell} \beta_{i,j}^{\square} \gamma_{i,\ell}(\square)(\underline{0})$. The cross section $s : F \to \mathrm{Y}$ will be defined as $x \mapsto \prod_{i,j,\ell} {\beta'}_{i,j}^{\square} \gamma'_{i,\ell}(\square)(y_0)$ where $q(y_0) = \underline{0}$.

Main theorem: CFR nilspaces are quotients of free nilspaces

Let us now prove one of the main results of the paper, that any CFR nilspace is a quotient of a free nilspace by a fiber-transitive group action.

Instead of showing first that the quotient by a fiber-transitive group action gives a nilspace, we are going to try to prove directly that any CFR nilspace is the quotient of a free nilspace by some subgroup of translations. This way we will see how the definition of fiber-transitive group action naturally appears.

Proof of main theorem (part 1)

The proof is going to be by induction on the step k of the $\textsc{cfr}$ nilspace X. The case $k = 1$ follows from abelian Lie group theory.

Step 1: Assume by induction that X_{k-1} equals the quotient of a $k-1$-step free nilspace F_{k-1} by the action of some finitely-generated group $H \leq \Theta(F_{k-1})$. The quotient map being $\varphi_{k-1} : F_{k-1} \to \mathrm{X}_{k-1}$.

Step 2: Consider the fiber-product
$F_{k-1} \times_{\mathrm{X}_{k-1}} \mathrm{X} := \{(f, x) \in F_{k-1} \times \mathrm{X} : \varphi_{k-1}(f) = \pi_{k-1}(x)\}$,

$$\begin{array}{ccc} F_{k-1} \times_{\mathrm{X}_{k-1}} \mathrm{X} & \xrightarrow{\;p_2\;} & \mathrm{X} \\ \big\downarrow p_1 & & \big\downarrow \pi_{k-1} \\ F_{k-1} & \xrightarrow{\;\varphi_{k-1}\;} & \mathrm{X}_{k-1}\,. \end{array}$$

Proof of main theorem (part 2)

Step 3: Show that $F_{k-1} \times_{\mathsf{X}_{k-1}} \mathsf{X}$ is an extension of F_{k-1} by $\mathsf{Z}_k(\mathsf{X})$, the last structure group of X. Hence $F_{k-1} \times_{\mathsf{X}_{k-1}} \mathsf{X}$ is isomorphic to $F_{k-1} \times \mathcal{D}_k(\mathsf{Z}_k(\mathsf{X}))$.

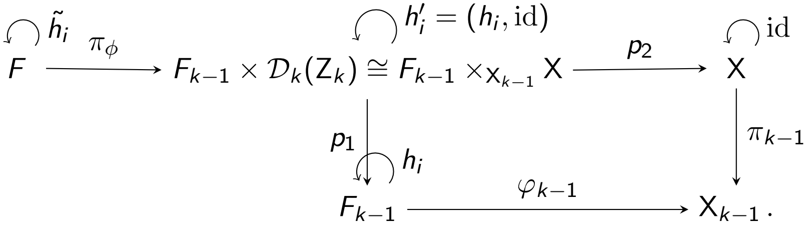
Step 4: There exists a group $\mathsf{Z} = \mathbb{Z}^r \times \mathbb{R}^s$ and a surjective homomorphism $\phi : \mathsf{Z} \to \mathsf{Z}_k(\mathsf{X})$. Thus if $F = F_{k-1} \times \mathcal{D}_k(\mathsf{Z})$ we can further refine the previous picture to:

$$
\begin{array}{ccccc}
F & \xrightarrow{\pi_\phi} & F_{k-1} \times \mathcal{D}_k(\mathsf{Z}_k) \cong F_{k-1} \times_{\mathsf{X}_{k-1}} \mathsf{X} & \xrightarrow{p_2} & \mathsf{X} \\
 & & \downarrow p_1 & & \downarrow \pi_{k-1} \\
 & & F_{k-1} & \xrightarrow{\varphi_{k-1}} & \mathsf{X}_{k-1}\,.
\end{array}
$$

Proof of main theorem (part 3)

Step 5: If $H = \langle h_1, \ldots, h_n \rangle$ note that we can lift these translations through the fibration $p_1 : F_{k-1} \times_{\mathsf{X}_{k-1}} \mathsf{X} \to F_{k-1}$ simply by taking $h_i' := (h_i, \mathrm{id})$ for $i \in [n]$. That is, on $F_{k-1} \times_{\mathsf{X}_{k-1}} \mathsf{X}$ we can define the translation that acts as h_i in the first coordinate and as id in the second.

Step 6: For every $i \in [n]$ we have that h_i' can be seen as a translation in $F_{k-1} \times \mathcal{D}_k(\mathsf{Z}_k(\mathsf{X}))$. But this is a group nilspace very explicit, and we can show that all translations here can be represented with polynomial maps. Moreover, using this explicit expression it can be shown that for each $i \in [n]$ there exists $\tilde{h}_i \in \Theta(F)$ such that $\pi_\phi \circ \tilde{h}_i = h_i' \circ \pi_\phi$.



Proof of main theorem (part 4)

Step 7: Recall that we took a surjective homomorphism $\phi : \mathsf{Z} \to \mathsf{Z}_k(\mathsf{X})$. We can do it in a way that the kernel is a lattice generated by $\gamma_1, \ldots, \gamma_m \in \mathsf{Z}$. For every $j \in [m]$ let $\tilde{\gamma}_j \in \Theta_k(F)$ be the map that adds γ_j to the Z component of $F = F_{k-1} \times \mathcal{D}_k(\mathsf{Z})$.

Step 8: Putting everything together the best candidate group H' in $\Theta(F)$ such that $\mathsf{X} \cong F/H'$ is precisely $H' := \langle \tilde{h}_1, \ldots, \tilde{h}_n, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_m \rangle$ and in fact this works. Moreover, this group satisfies the fiber-transitive property.

Thank you for your attention!