Disintegrating Cubic Measures for Sumsets

Donald Robertson
University of Manchester

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Work With

Bryna Kra (Northwestern)

Joel Moreira (Warwick University)

Florian K. Richter (EPFL)

Main Results

Theorem. If $A \subset \mathbb{N}$ has positive upper density then

$$B + C \subset A$$

for infinite sets $B, C \subset \mathbb{N}$.

Theorem. If $A \subset \mathbb{N}$ has positive upper density then

$$B_1 + B_2 + \cdots + B_k \subset A$$

for infinite sets $B_1, B_2, \ldots, B_k \subset \mathbb{N}$.

Dynamical Setup

$T : X \to X$ a homeomorphism of a compact metric space

μ a T-invariant and T-ergodic probability on X

$E \subset X$ clopen with $\mu(E) > 0$ and $A = \{n \in \mathbb{N} : T^n(a) \in E\}$

$a \in X$ generic for μ that is

$$\lim_{N \to \infty} \frac{1}{K_N} \sum_{n=J_N}^{J_N+K_N} \delta_{T^n a} = \mu$$

Dynamical Version of Main Result

Theorem (Main Theorem). There are sequences $b(n) \nearrow \infty$ and $c(n) \nearrow \infty$ and points $x, y, z \in X$ with

$$(T \times T)^{b(n)}(a, x) \to (y, z)$$
$$(T \times T)^{c(n)}(a, y) \to (x, z)$$

and $z \in E$.

Special Circumstances

If a belongs to the support of μ then one can find $b(n) \nearrow \infty$ with

$$T^{b(n)}(a) \to a$$

and take $c(n) = b(n) + t$ where $T^t(a) \in E$.

$$(T \times T)^{b(n)}(a, T^t(a)) \to (a, T^t(a))$$
$$(T \times T)^{b(n)+t}(a, a) \to (T^t(a), T^t(a))$$

Special Circumstances

In fact, if a belongs to the support of μ then A contains a translate of an IP set.

Example (Straus). There are $A \subset \mathbb{N}$ with positive density containing no translate of any IP set.

$$A = \mathbb{N} \setminus \bigcup_{n=0}^{\infty} \alpha_n(\mathbb{N} + \beta_n) + n$$

Limiting Configurations

A tuple (w, x, y, z) with sequence $b(n), c(n) \nearrow \infty$ satisfying

$$(T \times T)^{b(n)}(w, x) \to (y, z)$$
$$(T \times T)^{c(n)}(w, y) \to (x, z)$$

is called an Erdős cube.

Theorem (Main Theorem). An Erdős cube exists with first coordinate a and last coordinate in E.

Cubic Measures

Theorem. $\mu^{[\![2]\!]}$-almost every point in $X^{[\![2]\!]} = X^4$ is an Erdős cube.

Proof

1. Recall
$$\mu^{[\![2]\!]} = \int \lambda_{(p,q)} \times \lambda_{(p,q)} \, \mathrm{d}(\mu \times \mu)(p,q)$$
where $(p,q) \mapsto \lambda_{(p,q)}$ is an ergodic decomposition of $\mu \times \mu$ with respect to $T \times T$.
2. The map $(w,x,y,z) \mapsto (w,y,x,z)$ is a symmetry of $\mu^{[\![2]\!]}$.

Disintegration

The set $X^{[\![2]\!]}$ is partitioned by the fibers of the projection to the first coordinate

$$\pi_{00}(w, x, y, z) = w$$

and for each $w \in X$ we can attempt to determine the piece of $\mu^{[\![2]\!]}$ on $\pi_{00}^{-1}(w)$.

Disintegration

Want for every $w \in X$ a probability σ_w on $\pi_{00}^{-1}(w)$ such that

$$\int \sigma_w \, \mathrm{d}\mu(w) = \mu^{[\![2]\!]}$$

and to deduce that σ_a almost-every point is an Erdős cube.

Cubic Measure Again

It is standard that

$$\begin{aligned}\mu^{[\![2]\!]} &= \int \lambda_{(w,x)} \times \lambda_{(w,x)} \, \mathsf{d}(\mu \times \mu)(w,x) \\ &= \int \delta_{(w,x)} \times \lambda_{(w,x)} \, \mathsf{d}(\mu \times \mu)(w,x) \\ &= \int \delta_w \times \int \delta_x \times \lambda_{(w,x)} \, \mathsf{d}\mu(x) \, \mathsf{d}\mu(w)\end{aligned}$$

so we want to define $\sigma_w = \delta_w \times \int \delta_x \times \lambda_{(w,x)} \, \mathsf{d}\mu(x)$ for all $w \in X$.

Getting Main Theorem

1. Suppose $\sigma_w = \delta_w \times \int \delta_x \times \lambda_{(w,x)} \, \mathrm{d}\mu(x)$ makes sense for all w.
2. If σ_a has $(w, x, y, z) \mapsto (w, y, x, z)$ symmetry it suffices to prove for almost-every x that (a, x) is generic for $\lambda_{(a,x)}$.
3. Almost-every σ_t has (t, x) generic for $\lambda_{(t,x)}$ almost-surely.
4. If $w \mapsto \sigma_w$ is continuous and equivariant can find $s(n) \nearrow \infty$ with

$$(T \times T \times T \times T)^{s(n)} \sigma_a \to \sigma_t$$

Disintegration

Theorem (Disintegration Theorem).
If we have for every $w \in X$ a probability σ_w on $X^{[[2]]}$ with the properties

1. $\sigma_w(\{(w, x, y, z) : x, y, z \in X\}) = 1$
2. $w \mapsto \sigma_w$ continuous
3. $(T \times T \times T \times T)\sigma_w = \sigma_{T(w)}$
4. $w \mapsto \sigma_w$ disintegrates $\mu^{[\![2]\!]}$
5. $\pi_{11}\sigma_w = \mu$
6. σ_w invariant for $(w, x, y, z) \mapsto (w, y, x, z)$

then there is an Erdős cube (w, x, y, z) with $w = a$ and $z \in E$.

Disintegration

By standard results in measure theory we can get σ_w defined for μ almost-every w with

1. $\sigma_w(\{(w, x, y, z) : x, y, z \in X\}) = 1$
2. $w \mapsto \sigma_w$ measurable
3. $(T \times T \times T \times T)\sigma_w = \sigma_{T(w)}$
4. $w \mapsto \sigma_w$ disintegrates $\mu^{[\![2]\!]}$
5. $\pi_{11}\sigma_w = \mu$
6. σ_w invariant for $(w, x, y, z) \mapsto (w, y, x, z)$

But we are disintegrating a special measure!

Ergodic Decomposition

To define

$$\sigma_w = \delta_w \times \int \delta_x \times \lambda_{(w,x)} \, \mathrm{d}\mu(x)$$

for all $w \in X$ we need $\lambda_{(w,x)}$ to be defined for all (w, x) and for

$$(w, x) \mapsto \lambda_{(w,x)}$$

to be continuous.

Ergodic Decomposition

Non-Theorem There is a continuous map $(w, x) \mapsto \lambda_{(w,x)}$ from $X \times X$ to the space of probability measures on $X \times X$ with the following property: for $\mu \times \mu$ almost-every (w, x) the measure $\lambda_{(w,x)}$ is $T \times T$ invariant and ergodic.

Ergodic Decomposition

Where does the ergodic decomposition of $\mu \times \mu$ come from? It is controlled by the maximal compact group rotation factor (Z_1, m, R) of (X, μ, T).

$$\int f \otimes g \, \mathrm{d}\lambda_{(w,x)} = \int \mathbb{E}(f|Z_1) \otimes \mathbb{E}(g|Z_1) \, \mathrm{d}\lambda_{(w,x)}$$

We think of $\lambda_{(w,x)}$ as determined by the orbit of $(\pi w, \pi x)$ in $Z_1 \times Z_1$.

Ergodic Decomposition

Theorem. If there is a topological factor map $\pi : (X, T) \to (Z, R)$ realizing the Kronecker factor of (X, μ, T) then there is a continuous map $(w, x) \mapsto \lambda_{(w,x)}$ from $X \times X$ to the space of probability measures on $X \times X$ with the following property: for $\mu \times \mu$ almost-every (w, x) the measure $\lambda_{(w,x)}$ is $T \times T$ invariant and ergodic.

Proof

1. The map $\pi \times \pi : X \times X \to Z \times Z$ is continuous, and orbit closures vary continuously in $Z \times Z$. Thus the ergodic decomposition of $m \times m$ is continuously varying.
2. The structure theory allows us to lift this to a decomposition for $T \times T$ on $X \times X$ by associating with (w, x) the ergodic measure corresponding to the orbit of $(\pi w, \pi x)$.

Additional Properties

1. $\lambda_{(w,x)}$ only depends on $(\pi w, \pi x)$.
2. $(T \times T)\lambda_{(w,x)} = \lambda_{(Tw,Tx)}$ everywhere.
3. Every $\lambda_{(w,x)}$ is a self-joining of (X, μ, T).

Topological Factor Maps for Distal Factors

Theorem (Host Kra 2009). By passing to a topological extension, we can assume there is a factor map in the topological category to any given distal factor of (X, μ, T). Moreover, generic points survive the passage.

General Disintegration

Theorem. For every $k \in \mathbb{N}$ the inductive construction

$$\sigma_w^{[[1]]} = \delta_w \times \mu$$
$$\sigma_w^{[[k+1]]} = \int_{???} \delta_x \times \lambda_x^{[[k]]} \, \mathrm{d}\sigma_w^{[[k]]}(x)$$

is a disintegration of $\mu^{[[k+1]]}$ over $\pi_{00\ldots0}(x) = x_{00\ldots0}$ with all desirable properties.

Complications

For all this to work, we need a continuous ergodic decomposition of $\mu^{[\![k]\!]}$.

1. Orbit closures in powers of nilmanifolds do not foliate as nicely as orbit closures in compact abelian groups.
2. Our continuous ergodic decomposition of $\mu^{[\![k]\!]}$ is not defined on all of $X^{[\![k]\!]}$. It is only defined on an appropriate set of cubes.

Orbit Closures in Nilmanifolds

Theorem (Leibman 2009). In a connected nilmanifold Z there is a subnilmanifold Y with the following property: almost-every orbit closure $\overline{\{R^n(z) : n \in \mathbb{Z}\}}$ is a translate of Y.

Nilcubes

Let $\pi_k : X \to Z_k$ be a continuous factor map to the k-step pronilfactor of (X, μ, T).

Define the nilcubes of X to be

$$\mathrm{N}^{[\![k]\!]}(X) = (\pi_k^{[\![k]\!]})^{-1}(\mathrm{Q}^{[\![k]\!]}(Z_k))$$

where $\mathrm{Q}^{[\![k]\!]}(Z_k)$ are the (Host Kra Maass 2010) dynamical cubes.

Nilcubes

Our continuous ergodic decomposition of $\mu^{[\![k]\!]}$ is defined on $\mathsf{N}^{[\![k]\!]}(X)$.

One has $\mathsf{E}^{[\![k]\!]}(X) \subset \mathsf{Q}^{[\![k]\!]}(X) \subset \mathsf{N}^{[\![k]\!]}(X)$ where E denotes Erdős cubes.

In general $\mathsf{N}^{[\![k]\!]}(X)$ is larger than the support of $\mu^{[\![k]\!]}$ because *a* might not be in the support of μ.

Dziekuje!