Separating Recurrence Properties

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Będlewo

Let G be a countable abelian group.

d^* denotes upper Banach density in G.

Given a set $A \subset G$ with $d^*(A) > 0$, we want to understand the difference set $A - A := \{a - a' : a, a' \in A\}$.

Definition

Let $d \in \mathbb{N}$, $\rho : G \to \mathbb{T}^d$ a homomorphism, and $U \subset \mathbb{T}^d$ an open neighborhood of $0 \in \mathbb{T}^d$. The Bohr neighborhood in G determined by these parameters is $\rho^{-1}(U)$.

A Bohr neighborhood of $g \in G$ has the form $g + B$, where B is a Bohr neighborhood of 0. The Bohr topology on G is the smallest topology containing all Bohr neighborhoods.

Theorem (Følner 1954)

If $A \subset G$ and $d^*(A) > 0$, then $A - A$ contains $B \setminus E$, where B is a Bohr neighborhood of 0 in G and $d^*(E) = 0$.

Ruzsa asked [Ruz82]: if $d^*(A) > 0$, must $A - A$ contain a Bohr neighborhood of 0? (Can E be eliminated in Følner's thm?)

Lack of structure in $A - A$

Kriz (1987) showed that there are sets $A \subset \mathbb{Z}$ with $d^*(A) > 0$ where $A - A$ does not contain a Bohr neighborhood of 0.

Problem

Classify the sets $A \subset \mathbb{Z}$ with $d^*(A) > 0$ such that $A - A$ does not contain a Bohr neighborhood of 0.

Ruzsa [Ruz85], Ruzsa [GR09], G. [Gri12] Bergelson and Ruzsa [BR09], Hegyvári and Ruzsa [HR16] asked:

Question

If $A \subset \mathbb{Z}$ and $d^*(A) > 0$, must $A - A$ contain a Bohr neighborhood of some $n \in \mathbb{Z}$?

G. [Gri21]: no.

Problem

Classify the sets $A \subset \mathbb{Z}$ with $d^*(A) > 0$ such that $A - A$ does not contain a Bohr neighborhood of any $n \in \mathbb{Z}$.

$A - A$ contains no Bohr nhood - a finite analogue

$\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

In $\mathbb{F}_2^d$, $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc.

$\mathcal{E} = \{e_i : i = 1, \ldots, d\} \subset \mathbb{F}_2^d$.

Subgroups (= subspaces) play the role of Bohr nhoods in $\mathbb{F}_2^d$.

Lemma

Fix $K \in \mathbb{N}$ and $\delta < 1/2$. For all large enough d, there is an $A \subset \mathbb{F}_2^d$ with $|A| > \delta 2^d$ such that $A - A$ does not contain any coset of any subspace of codimension K.

Setup: for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{F}_2^d$, $w(\mathbf{x}) = |\{i \le d : x_i \neq 0\}|$.

The Hamming ball of radius k around $\mathbf{y} \in \mathbb{F}_2^d$ is

$$H(\mathbf{y}; k) := \{\mathbf{x} \in \mathbb{F}_2^d : w(\mathbf{x} - \mathbf{y}) \le k\}$$

Note that $H(\mathbf{0}; k) = \{0\} \cup \mathcal{E} \cup (\mathcal{E} + \mathcal{E}) \cup \cdots \cup (\sum_{i=1}^{k} \mathcal{E})$

Claim (in $\mathbb{F}_2^d$)

(i) *Let* $S_K := H(\mathbf{1}; K)$*. Then* $S_K \cap V \neq \varnothing$ *when* V *is a coset of a subspace of codimension* K*.*

(ii) *For* $\delta < \frac{1}{2}$ *and large enough* d*,* $A_K := H(\mathbf{0}; \frac{d}{2} - K) > \delta|\mathbb{F}_2^d|$*.*

(iii) $(A_K - A_K) \cap S_K = \varnothing$*.*

(i) and (iii) $\implies$ $A_K - A_K$ contains no coset of a subspace of codimension K.

Proof of (i). The property is translation invariant, so it suffices to prove the statement for $H(\mathbf{0}; K)$ instead.

Equivalent to: if $\rho : \mathbb{F}_2^d \to \mathbb{F}_2^K$ is surjective, then $\rho(H(\mathbf{0}; K)) = \mathbb{F}_2^K$.

But $\rho(\mathcal{E})$ spans $\mathbb{F}_2^K$, and

$$\rho(H(\mathbf{0}; K)) = \rho(\mathbf{0}) \cup \rho(\mathcal{E}) \cup \rho(\mathcal{E} + \mathcal{E}) \cup \cdots \cup \rho(\sum_{i=1}^{K} \mathcal{E}) \qquad \square$$

IS THERE A FUNDAMENTALLY DIFFERENT WAY TO CONSTRUCT DENSE SETS WHOSE DIFFERENCE SETS LACK STRUCTURE?

Let G be a countable abelian group. $A \subset G$ is syndetic if there is a finite set F such that $A + F = G$. (So $d^*(A) \geq 1/|F|$).

Question (Veech [Vee68], Landstad [Lan71], Ruzsa [Ruz85], Glasner [Gla98], Katznelson [Kat01])

If $A \subset G$ is syndetic, must $A - A$ contain a Bohr neighborhood of 0?

The answer is unknown in every countably infinite abelian group G.

Problems on difference sets are equivalent to problems about sets of recurrence in various categories of dynamical systems.

If $d^*(A) > 0$, there is a probability measure preserving G-system (X, μ, T) and $\tilde{A} \subset X$ with $\mu(\tilde{A}) = d^*(A)$ such that $A - A$ contains $\{g \in G : \tilde{A} \cap T_g^{-1}\tilde{A} \neq \varnothing\}$.

If A is syndetic there is a minimal topological G-system (X, T) and an open $U \subset X$ such that $A - A$ contains $\{g : U \cap T_g^{-1}U \neq \varnothing\}$, We want to understand single recurrence in various categories of dynamical systems.

Definition

Let G be a countable abelian group and $S \subset G$.

- S is a set of measurable recurrence if for every probability measure preserving G-system (X, μ, T) and every $A \subset X$ with $\mu(A) > 0$, there is a $g \in S$ such that $A \cap T_g^{-1} A \neq \varnothing$.
- S is a set of topological recurrence if for every minimal topological G-system (X, T) and every nonempty open $A \subset X$, there is a $g \in S$ such that $A \cap T_g^{-1} A \neq \varnothing$.
- S is a set of Bohr recurrence if for every minimal group rotation system (Kronecker system) (K, T) and every open $A \subset K$, there is a $g \in S$ such that $A \cap T_g^{-1} A \neq \varnothing$.

S is a set of measurable rec. $\implies$ S is a set of top. rec. $\implies$ S is a set of Bohr rec.

Sets of measurable recurrence: examples

$S \subset G$ is a set of measurable recurrence if $\forall$ G-systems (X, μ, T) and $A \subset X$ with $\mu(A) > 0$, $\exists\, g \in S$ such that $A \cap T_g^{-1}A \neq \varnothing$.

Examples:

- In any group: if $E \subset G$ is infinite, then $\{a - b : a \neq b \in E\}$ is a set of measurable recurrence. (Poincaré)
- In $\mathbb{Z}$: $\{n^2 : n \in \mathbb{N}\}$ (Furstenberg, Sárközy)
- $\{\lfloor n^{5/2} \rfloor : n \in \mathbb{N}\}$

Non-examples:

- $\{n^2 + 1 : n \in \mathbb{N}\}$ is not a set of measurable recurrence. Also not a set of Bohr recurrence.
- $\{n! : n \in \mathbb{N}\}$ is not a set of measurable recurrence. Also not a set of Bohr recurrence.

Is there some common structure underlying all sets of measurable recurrence? Something common to all sets of topological recurrence? Bohr recurrence?

Kriz's example

- $S \subset G$ is a set of measurable recurrence if $\forall$ G-systems (X, μ, T) and $A \subset X$ with $\mu(A) > 0$, $\exists$ $g \in S$ such that $A \cap T_g^{-1}A \neq \varnothing$.
- S is a set of topological recurrence if $\forall$ min. top. G-systems (X, T) and open $A \subset X$ $\exists$ $g \in S$ such that $A \cap T_g^{-1}A \neq \varnothing$.

(Early '80s) Bergelson, Furstenberg, and Ruzsa asked: is every set of topological recurrence a set of measurable recurrence?

Theorem (Kriz [Kri87])

In $\mathbb{Z}$, there is a set of topological recurrence which is not a set of measurable recurrence.
Moreover: if $E \subset \mathbb{Z}$ is infinite, then there is a subset of $E - E$ which is a set of topological but not measurable recurrence.

The second statement is implicit in [Kri87]. Proved explicitly in [Gri23a] "Separating topological recurrence from measurable recurrence".

Classifying Kriz-type examples

Can we classify the systems that separate topological recurrence for measurable recurrence?

What can we say about an ergodic MPS (X, μ, T) admitting an A with $\mu(A) > 0$ and $A \cap T^{-n}A = \varnothing$ for all n in a set of topological recurrence? Bohr recurrence?

Question

Is every set of topological recurrence also a set of measurable recurrence for distal systems (i.e. inverse limits of compact extensions)?

(I believe the Kriz example can be done with (X, μ, T) weak mixing.)

Problem (Chandgotia and Weiss 2020)

Prove or disprove: if S is a set of measurable recurrence for all zero entropy measure preserving systems then S is a set of measurable recurrence.

Measurable vs. topological recurrence in other groups

$\mathbb{F}_p^\omega$ is the countably infinite vector space over $\mathbb{F}_p$.

Alan Forrest [For91] constructed examples of topological but not measurable recurrence in $\mathbb{F}_2^\omega$.

Such examples can be lifted from subgroups and quotients.

Question

Let p be an odd prime. Is there a subset of $\mathbb{F}_p^\omega$ with which is a set of topological but not measurable recurrence?

Hereditary separation

Question

Let $S \subset \mathbb{Z}$ be a set of measurable recurrence. Must there be a set $S' \subset S$ which is a set of topological recurrence and not measurable recurrence?

This is true for $S = \{n^2 : n \in \mathbb{N}\}$ and other familiar, explicit examples. Can be done by modifying Kriz's construction.

[Gri23b] constructs a set $S \subset \{n^2 : n \in \mathbb{N}\}$ which is a set of Bohr recurrence but not measurable recurrence.

But for a completely arbitrary set of recurrence S, new techniques may be required.

The same question makes sense for any other pair of recurrence properties.

Katznelson's problem

- $S \subset G$ is a set of topological recurrence if $\forall$ min. top. G-systems (X, T) and open $A \subset X$ $\exists$ $g \in S$ such that $A \cap T_g^{-1} A \neq \varnothing$.
- S is a set of Bohr recurrence if for all min. Kronecker systems (K, T) and open $A \subset K$ $\exists$ $g \in S$ such that $A \cap T_g^{-1} A \neq \varnothing$.

Veech [Vee68] asked: is every set of Bohr recurrence a set of topological recurrence?

Reiterated by Landstad [Lan71], Ruzsa [Ruz82], Glasner [Gla98], Katznelson [Kat01].

Known special cases

Katznelson's problem: is every set of Bohr recurrence also a set of topological recurrence?

Kunen and Rudin [KR99]: if $E \subset \mathbb{Z}$ is lacunary and $S \subset E - E$ is a set of Bohr recurrence, then S is a set of topological recurrence. Extended to countable abelian groups in [Gri23c].

Note: $E - E$ is a set of measurable recurrence, and (by Kriz) contains a set of topological but not measurable recurrence.

Host, Kra, Maass [HKM16]: if S is a set of Bohr recurrence, then S is a set of recurrence for minimal nilsystems.

Glasscock, Koutsogiannis, Richter [GKR22]: if S is a set of Bohr recurrence, then S is a set of recurrence for a special class of minimal skew product systems.

Examples with unknown recurrence properties

Given a countable abelian group G, $\varepsilon > 0$, a homomorphism $\rho : G \to \mathbb{T}^d$, and a neighborhood U of 0 in $\mathbb{T}^d$, the Bohr neighborhood of 0 determined by these parameters is

$$\{g \in G : \rho(g) \in U\}.$$

S is a set of Bohr recurrence if and only if $S \cap B \neq \varnothing$ for every Bohr neighborhood of 0.

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}, \qquad \mathbb{F}_p^\omega = \mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_p \oplus \cdots$$

The Bohr neighborhoods of 0 in $\mathbb{F}_p^\omega$ are the finite index subgroups.

Observation

$S \subset \mathbb{F}_p^\omega$ is a set of Bohr recurrence iff for every $d \in \mathbb{N}$ and every homomorphism $\rho : \mathbb{F}_p^\omega \to \mathbb{F}_p^d$, $\exists g \in S$ such that $\rho(g) = 0$.

Bohr recurrence in $\mathbb{F}_3^\omega$

$e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots,)$, $\dots$
$\mathcal{E} = \{e_i : i \in \mathbb{N}\}$ is the standard basis of $\mathbb{F}_3^\omega$.

$\mathcal{E}$ is not a set of Bohr recurrence

$\rho : \mathbb{F}_3^\omega \to \mathbb{F}_3$, $\rho((x_1, x_2, \dots,)) := \sum x_i$ maps $\mathcal{E}$ to $\{1\}$.

$\mathcal{E}_3 := \{e_i + e_j + e_k : i, j, k \text{ mutually distinct}\}$
$\mathcal{E}_3$ is a set of Bohr recurrence.
Proof. If $\rho : \mathbb{F}_3^\omega \to \mathbb{F}_3^d$ is a homomorphism, choose i, j, k so that $\rho(e_i) = \rho(e_j) = \rho(e_k)$. Then

$$\rho(e_i + e_j + e_k) = \rho(e_i) + \rho(e_j) + \rho(e_k) = 3\rho(e_i) = 0.$$

Which subsets of $\mathcal{E}_3$ are sets of Bohr recurrence?
Let $\mathcal{F}$ be a collection of subsets of $\mathbb{N}$. Say $\mathcal{F}$ is partition regular if for every partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, $F \subset C_i$ for some $F \in \mathcal{F}$, $i \leq r$.

Lemma (Givens [Giv03], Givens and Kunen [GK03])

Let $\mathcal{F}$ be a collection of 3-element subsets of $\mathbb{N}$, let

$$\mathcal{E}_{\mathcal{F}} := \{e_i + e_j + e_k : \{i, j, k\} \in \mathcal{F}\} \subset \mathbb{F}_3^{\omega}$$

$\mathcal{E}_{\mathcal{F}}$ is a set of Bohr recurrence if and only if $\mathcal{F}$ is partition regular.

$S_{3AP} := \{e_n + e_{n+d} + e_{n+2d} : n, d \in \mathbb{N}\}$

$S_{Schur} := \{e_a + e_b + e_{a+b} : a, b \in \mathbb{N}\}$

Are these sets of topological recurrence? Measurable recurrence?

Proposition (G. [Gri23c])

If every subset of $\mathcal{E}_3$ which is a set of Bohr recurrence is a set of topological recurrence, then every subset of $\mathbb{F}_3^{\omega}$ which is a set of Bohr recurrence is a set of topological recurrence.

So we only need to look among subsets of $\mathcal{E}_3$ to resolve Katznelson's question for $\mathbb{F}_3^{\omega}$.

One more example

$$\{k!2^n3^m : k, n, m \in \mathbb{N}\}$$

is a set of Bohr recurrence in $\mathbb{Z}$
(asserted by Frantzikinakis and McCutcheon 2009).

Is it a set of topological recurrence? Measurable recurrence?

Translates with recurrence properties

A sequence $(g_n)_{n\in\mathbb{N}}$ in G is Hartman uniformly distributed if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi(g_n)=0.$$

for every nontrivial homomorphism $\chi : G \to \mathbb{S}^1$ (= unit circle in $\mathbb{C}$).
In $\mathbb{Z}$ this means $\frac{1}{N}\sum_{n=1}^{N}\exp(g_n t)\to 0$ for every $t\in(0,2\pi)$.

Hartman-u.d. implies $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T_{g_n}^{-1}A)\geq\mu(A)^2$.

So if a sequence of elements of S is Hartman-u.d., then S is a set of measurable recurrence.
Every translate (g_n+t) of a Hartman-u.d. sequence is Hartman-u.d.
So every translate of S is a set of measurable recurrence (and therefore a set of topological recurrence and a set of Bohr recurrence).

Katznelson [Kat73] constructed a set $S \subset \mathbb{Z}$ where every translate of S is a set of Bohr rec., but no sequence from S is Hartman-u.d.

Construction: define $a_1 = 1$, $a_{n+1} = na_n + 1$.
$S :=$ the IP set generated by a_n: $S = \{\sum_{n \in F} a_n : F \subset \mathbb{N} \text{ finite}\}$.

Katznelson proved:

(i) Every translate of S is a set of Bohr recurrence.

(ii) There is a continuous probability measure σ on $\mathbb{T}$ such that $|\hat{\sigma}(n)| > 1 - \varepsilon$ for every element of S. This is impossible for Hartman-u.d. sequences.

S is a set of measurable recurrence (it's an IP set).
$S + 1$ is a set of Bohr recurrence.

Question

Is $S + 1$ a set of topological recurrence? Measurable recurrence?

Are there analogues of this construction in other groups?

Badea, Grivaux, and Matheron [BGM19] refined Katznelson's example to construct rigidity sequences with interesting properties.

Saeki [Sae80], G. [Gri19], Ackelsberg [Ack22] studied recurrence properties via Fourier transforms of measures on $\widehat{G}$.

Theorem ([Gri18])

(In $\mathbb{Z}$) If every translate of S is a set of measurable recurrence, then there is a set $S' \subset S$ such that every translate of S' is a set of measurable recurrence, and no sequence from S' is Hartman-u.d.

There seems to be some nontrivial structure common to sets S where every translate of S is a set of measurable recurrence. Is there some structure common to all sets of measurable recurrence?

More translates and recurrence

Ruzsa ([Ruz85], [GR09]), Bergelson and Ruzsa [BR09], G. [Gri12], Hegyvári and Ruzsa [HR16], asked (in $\mathbb{Z}$): if every translate of S is a set of Bohr recurrence, must S be a set of measurable recurrence?

G. [Gri21]: No. Moreover, if $S \subset \mathbb{Z}$ and every translate of S is a set of Bohr recurrence, then there is an $S' \subset S$ such that every translate of S' is a set of Bohr recurrence and S' is not a set of measurable recurrence.

There seems to be some nontrivial structure common to sets which are dense in the Bohr topology.

Theorem ([Kri87])

There is a set $S \subset \mathbb{Z}$ which is a set of topological recurrence but is not a set of measurable recurrence.

S is constructed from finite approximations. Here is a rough outline:

Definition

- $S \subset G$ is δ-nonrecurrent if $\exists\ B \subset G$ with $d^*(B) > \delta$ such that $B \cap (B + S) = \varnothing$.
- $S \subset G$ is k-chromatically recurrent if for every partition $G = B_1 \cup B_2 \cup \cdots \cup B_k$, we have $B_i \cap (B_i + S) \neq \varnothing$ for some B_i.
  Equivalently, the Cayley graph on G determined by S has chromatic number $> k$.

Lemma (gluing lemma)

In $\mathbb{Z}$: if $S_1, S_2 \subset \mathbb{Z}$ are finite, S_1 is δ_1-nonrecurrent and S_2 is δ_2-nonrecurrent, then for all sufficiently large m, $S_1 \cup mS_2$ is $2\delta_1\delta_2$-nonrecurrent.

It suffices to find, for each k, a finite set S_k which is $(\frac{1}{2} - \varepsilon_k)$-nonrecurrent and k-chromatically recurrent ($\varepsilon_k \to 0$ rapidly).

$S = S_0 \cup m_1 S_1 \cup m_2 S_2 \cup \cdots$

A compactness property of measurable recurrence says that S will be δ-nonrecurrent for any $\delta < (1 - 2\varepsilon_1)(1 - 2\varepsilon_2)\cdots$.

S will be k-chromatically recurrent for every k, so will be a set of topological recurrence.

These examples are essentially constructed in $\mathbb{F}_2^d$ and "copied" into $\mathbb{T}^d$, then into $\mathbb{Z}$.

Hamming balls in $\mathbb{F}_2^d$

For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{F}_2^d$, $w(\mathbf{x}) = |\{i : x_i \neq 0\}|$.

For $\mathbf{y} \in \mathbb{F}_2^d$, the Hamming ball around $\mathbf{y}$ of radius k is

$$H(\mathbf{y}; k) := \{\mathbf{x} \in \mathbb{F}_2^d : w(\mathbf{y} - \mathbf{x}) \leq k\}$$

$\mathbf{1} = (1, \dots, 1) \in \mathbb{F}_2^d$.

When $d > 2k$, $S_k := H(\mathbf{1}; 2k)$ is k-chromatically recurrent.

Reason: the Cayley graph determined by $H(\mathbf{1}; 2k)$ contains a copy of the Kneser Graph $KG(d, \frac{d}{2} - k)$, which has chromatic number $k + 2$ (Lovász).

When d is very large ($k = o(\sqrt{d})$), $H(\mathbf{1}; 2k)$ is δ-nonrecurrent ($\delta \to 1/2$ as $d \to \infty$.)

Reason: let $A = H(\mathbf{0}; \frac{d}{2} - 2k)$. A has density $\to \frac{1}{2}$ as $d \to \infty$. Then $A - A = H(\mathbf{0}; d - 4k)$, which is disjoint from $H(\mathbf{1}; 2k)$.

What about $\mathbb{F}_p^d$ for odd p?

We say $S \subset G$ is k-Bohr recurrent if $S \cap B \neq \varnothing$ for every Bohr neighborhood B of 0 determined by a homomorphism $\rho : G \to \mathbb{T}^k$. In $\mathbb{F}_p^d$ this means that $S \cap H \neq \varnothing$ whenever H is a subgroup of index p^k.

If $d > k$, then $H(\mathbf{1}; p \cdot k)$ is k-Bohr recurrent in $\mathbb{F}_p^d$.
(k-chromatically recurrent?)

But $H(\mathbf{1}; p \cdot k)$ is δ-nonrecurrent ($\delta \to \frac{1}{2} - \frac{1}{2p}$ as $d \to \infty$) [Gri21].

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