On Elliott's conjecture and applications

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joint work with Oleksiy Klurman and Alexander P. Mangerel

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- $\chi(n)n^{it}$ with χ a Dirichlet character and $t \in \mathbb{R}$ ($\chi(mn) = \chi(m)\chi(n)$ for all m, n and χ is periodic of some period q and $\chi(n) = 0$ if $\gcd(n, q) > 1$).

Mean values and correlations

Given a multiplicative function $g : \mathbb{N} \to \mathbb{D}$, we wish to understand the mean values

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} g(n)$$

and the correlations

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Introduce a distance function on the space of multiplicative functions (pretentious distance):

$$\mathbb{D}(f,g;x) := \left(\sum_{p\le x} \frac{1-\mathrm{Re}(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

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If g pretends to be g', their mean value and correlation behaviours should be similar.

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If instead g pretends to be 1, then the mean value exists and can be computed (Delange's theorem), but if g pretends to be n^{it} with $t \neq 0$, the limit usually does not exist.

Elliott's conjecture, 1990

Let $k \geq 1$, and let $g_1, \dots, g_k : \mathbb{N} \to \mathbb{D}$ be multiplicative functions. Let $h_1, \dots, h_k \in \mathbb{N}$ be distinct. Suppose that g_1 does not pretend to be $\chi(n)n^{it}$ for any Dirichlet character χ and $t \in \mathbb{R}$. Then

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- The case $g_1 = \cdots = g_k = \lambda$ is Chowla's conjecture.
- One needs to exclude functions pretending to be characters, since if $g(n) = \chi(n)n^{it}$ with $\chi \pmod q$ a character and $t \in \mathbb{R}$, then

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- If g_j pretends to be $\chi_j(n)n^{it_j}$ for $1 \leq j \leq k$, then there is an asymptotic formula for the correlations due to Klurman (2016).

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There is an interesting class of non-pretentious multiplicative functions that are not strongly non-pretentious (the MRT class), recently studied by Gomilko–Lemańczyk–de la Rue and Frantzikinakis–Lemańczyk–de la Rue.

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Let $k \geq 1$, and let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{D}$ be multiplicative functions. Let $h_1, \ldots, h_k \in \mathbb{N}$ be distinct. Suppose that g_1 does not pretend to be $\chi(n)n^{it}$ for any Dirichlet character χ and $t \in \mathbb{R}$. Then there exists some set $\mathcal{X} \subset \mathbb{N}$ of upper logarithmic density 1, such that

$$\lim_{\substack{x \to \infty \\ x \in \mathcal{X}}} \frac{1}{x} \sum_{n \leq x} g_1(n + h_1) \cdots g_k(n + h_k) = 0.$$

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Here, if $\mathcal{X} = \{x_1, x_2, \ldots\}$ with $x_1 < x_2 < \cdots$, then $\lim_{x \to \infty, x \in \mathcal{X}} f(x) := \lim_{n \to \infty} f(x_n)$.

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The upper logarithmic density is $\limsup_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x, n \in \mathcal{X}} 1/n$.

Main theorems

Theorem (Klurman–Mangerel–T., 2023)

Let $g : \mathbb{N} \to \mathbb{D}$ be multiplicative and let $h_1, \dots, h_k \in \mathbb{N}$ be distinct. Suppose that g does not pretend to be $\chi(n)n^{it}$ for any χ and t.
(i) We have

$$\lim_{\substack{x\to\infty \\ x\in\mathcal{X}}} \frac{1}{x}\sum_{n\leq x} g(n+h_1)\overline{g}(n+h_2) = 0$$

for some set $\mathcal{X} \subset \mathbb{N}$ of upper logarithmic density 1.

This strengthens earlier work of Tao (2016) and Tao–T. (2019), where for (i) we had to assume that g is strongly non-pretentious and for (ii) that $\limsup_{x\to\infty} \mathbb{D}(g,\chi;x)^2/\log\log x > 0$.

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Let $\mathcal{P} \subset \mathbb{P}$ be any subset of the primes of relative density 0 and such that $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$. Let $g(n) = (-1)^{\Omega_{\mathcal{P}}(n)}$. Then, for any distinct $h_1, \ldots, h_k \in \mathbb{N}$,

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As a consequence, it follows that $g : \mathbb{N} \to \mathbb{D}$ above satisfies Sarnak's conjecture: if (Y, T) is any topological dynamical system of zero entropy and $f : Y \to \mathbb{C}$ is continuous and $y_0 \in Y$, then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} g(n) f(T^n y_0) = 0.$$

Furstenberg systems of multiplicative functions

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Frantzikinakis and Host proved that these Furstenberg systems have as their ergodic components direct products of infinite-step nilsystems and Bernoulli systems.

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Theorem (Klurman–Mangerel–T., 2023)

Let $f : \mathbb{N} \to \{0, 1\}$ be a pretentious multiplicative function. Let ν be its unique Furstenberg measure. Then there is (an explicit) multiplicative function $g : \mathbb{N} \to \{-1, 0, +1\}$ whose unique Furstenberg system is isomorphic to the direct product of ν and a Bernoulli system.

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Bergelson–Kułaga-Przymus–Lemańczyk–Richter and Frantzikinakis–Lemańczyk–de la Rue give characterisations of Furstenberg systems of pretentious functions.

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Theorem (Schur, 1973)

There exists a collection $\mathcal{F}_3$ of 2 completely multiplicative functions such that the following holds. Let $g : \mathbb{N} \to \{-1, +1\}$ be completely multiplicative. Then $(g(n), g(n+1), g(n+2)) = (+1, +1, +1)$ for infinitely many n iff $g \notin \mathcal{F}_3$.

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Explicitly, $\mathcal{F}_3 = \{g_3^+, g_3^-\}$, where $g_3^\pm(p) = \chi_3(p)$ for $p \neq 3$ and $g_3^+(3) = +1$, $g_3^-(3) = -1$.

Hudson's conjecture

Conjecture (Hudson, 1974)

There exists a collection $\mathcal{F}_4$ of 13 completely multiplicative functions such that the following holds. Let $g : \mathbb{N} \to \{-1, +1\}$ be completely multiplicative. Then $(g(n), g(n+1), g(n+2), g(n+3)) = (+1, +1, +1, +1)$ for infinitely many n if and only if $g \notin \mathcal{F}_4$.

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Hudson's conjecture is true.

In fact, we prove that for $g \notin \mathcal{F}_4$ the pattern $++++$ is attained with positive lower density.

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The Erdős discrepancy theorem

In 2015, Tao used his logarithmic two-point Elliott conjecture to solve an old problem of Erdős:

Theorem (Tao)

Let $g : \mathbb{N} \to \{-1, +1\}$ be arbitrary. Then

$$\sup_{d,x \geq 1} \Big| \sum_{n \leq x} g(dn) \Big| = \infty.$$

In particular, if $g : \mathbb{N} \to \{-1, +1\}$ is completely multiplicative,

$$\sup_{x \geq 1} \Big| \sum_{n \leq x} g(n) \Big| = \infty.$$

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Theorem (Klurman–Mangerel–T., 2023)

Let $g : \mathbb{N} \to \{-1, +1\}$ be completely multiplicative and let $M \geq 1$. Then the set

$$\{x \in \mathbb{N} : \ \big|\sum_{n \leq x} g(n)\big| \geq M\}$$

has positive upper logarithmic density.

Proof ideas

We prove an upper bound for correlations under a weak non-pretentiousness hypothesis. Precisely, for any $\varepsilon > 0$ and $x \geq x_0(\varepsilon)$ we have

$$\Big|\frac{1}{x}\sum_{n\leq x} g_1(n+h_1)\cdots g_k(n+h_k)\Big| \leq \varepsilon,$$

provided that

$$\sum_{p\leq x}\frac{1-\mathrm{Re}(g_j(n)\overline{\chi_j}(n)n^{-it_j})}{p} \geq \frac{1}{\varepsilon}, \quad \sum_{x^{\varepsilon}\leq p\leq x}\frac{1-\mathrm{Re}(g_j(n)\overline{\chi_j}(n)n^{-it_j})}{p} \leq \varepsilon^3$$

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The proof uses some sieve theory and Euler product estimates.

Proof ideas

For the proof of the $k = 2$ case, we prove the following strengthening of the earlier result of Tao–T.: Let $A(X)$ be any function tending to ∞ and suppose that

$$\mathcal{S} := \{x \in \mathbb{N} : \inf_{|t| \le x} \min_{\substack{\chi \pmod q \\ q \le A(x)}} \mathbb{D}(g, \chi(n)n^{it}; x) \ge A(x)\}$$

is infinite. Then there is a set $\mathcal{X}$ of upper logarithmic density 1 such that

$$\lim_{\substack{x \to \infty \\ x \in \mathcal{X}}} \frac{1}{x} \sum_{n \le x} g(n + h_1)\overline{g}(n + h_2) = 0.$$

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We prove this by adapting Tao's work on the logarithmic two-point Elliott conjecture.

Proof ideas

Now, by our second auxiliary result, we can find the desired set of logarithmic density 1 unless for some χ_x and $|t_x| \le x$ we have

$$\mathbb{D}(g, \chi_x(n)n^{it_x}; x) \ll \log\log\log x.$$

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We can show, using the theory of pretentiousness, that this implies

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By a pigeonholing argument, this implies that, for any $\varepsilon > 0$, for an upper logarithmic density 1 of $x \in \mathbb{N}$ we have

$$\sum_{x^\varepsilon \leq p \leq x} \frac{1 - \mathrm{Re}(g(n)\overline{\chi}(n)n^{-it})}{p} \leq \varepsilon^3.$$

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But then by the first auxiliary result we have

$$\Big|\frac{1}{x}\sum_{n\leq x} g_1(n+h_1)\cdots g_k(n+h_k)\Big| \leq \varepsilon.$$

Letting $\varepsilon \to 0$ concludes the proof.

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Letting $\varepsilon \to 0$ concludes the proof. Thank you!