Mean ergodic theorems along sequences of polynomial growth

Konstantinos Tsinas

University of Crete

General setting

For any standard probability space $(X, \mathcal{X}, \mu)$, measure preserving map T, sequences $a_1(n), ..., a_k(n)$ of integers and functions $f_1, ..., f_k \in L^\infty(\mu)$, we want to study the L^2-limit of the averages

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Goal

Establish the mean convergence of multiple ergodic averages along other sparse subsequences of the integers, which do not grow faster than polynomials. When do these averages converge to the product of the integrals of the functions $f_1, ..., f_k$?

Sequences that we study

Example

Some examples that we want to study are:

- All integer polynomial sequences like $n^2, n^4 + n^3 + 2$. We can also include real polynomials using the integer part function, such as $\lfloor n\sqrt{2} \rfloor$ and $\lfloor \frac{n^3}{3} + n\sqrt{5} \rfloor$.
- Sequences of fractional powers like $\lfloor n^{3/2} \rfloor, \lfloor n^{\pi} \rfloor$.
- More generally, if $f : \mathbb{R} \to \mathbb{R}$ is a "well-behaved" function with $f(t) = O(t^k)$ for some $k \in \mathbb{N}$, we consider the sequence $\lfloor f(n) \rfloor$. Examples are the sequences $\lfloor n^2 \log n \rfloor$ and $\lfloor n \log n + e^{\sqrt{\log n}} \rfloor$.
- The function $f(t)$ has to be smooth, have polynomial growth and not oscillate substantially for large values of t for our methods to work.

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- The prototypical example is the field $\mathcal{LE}$ of logarithmico-exponential functions (Hardy 1912). These are functions defined by a finite combination of the basic algebraic operations and the functions exp and log acting on a real variable.

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- The prototypical example is the field $\mathcal{LE}$ of logarithmico-exponential functions (Hardy 1912). These are functions defined by a finite combination of the basic algebraic operations and the functions exp and log acting on a real variable.
- Examples of functions that do not belong in $\mathcal{H}$ are those that contain the trigonometric functions, such as the function $t \to t \sin t$.

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Functions in $\mathcal{H}$ behave "nicely" as $t \to +\infty$.

- For functions $f \in \mathcal{H}$, we have that the limit of f, as $t \to +\infty$, exists (possibly ∞).
- Functions in $\mathcal{H}$ are smooth and eventually monotone.
- Any two functions $f, g \in \mathcal{H}$ are comparable, meaning the limit of the ratio $f(t)/g(t)$ at $+\infty$ exists. Therefore, it makes sense to compare the growth rates of any two functions in $\mathcal{H}$.

The $k=1$ case

- Assume that a function $a \in \mathcal{H}$ has polynomial growth. What can we say about the single averages

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- Using the spectral theorem, we can find a bounded measure v on the torus, such that $\int \bar{f} \cdot T^n f \; d\mu = \int_{\mathbb{T}} \; e(na) \; dv(a)$. Then, we have

$$\Big\| \frac{1}{N}\sum_{n=1}^{N} T^{\lfloor a_1(n) \rfloor} f \Big\|_{L^2(\mu)}^2 = \int_{\mathbb{T}} \Big| \frac{1}{N}\sum_{n=1}^{N} e(\lfloor a_1(n) \rfloor a) \Big|^2 dv(a).$$

  Thus, we need to study equidistribution (mod 1) of the sequence $\lfloor a_1(n) \rfloor a$ for $a \in [0, 1)$.

The $k = 1$ case

Boshernitzan-1994

Assume the function $a \in \mathcal{H}$ has polynomial growth. Then, the sequence $a(n)$ is uniformly distributed (mod 1) iff for every polynomial $p \in \mathbb{Q}[t]$, we have

$$\lim_{t \to +\infty} \frac{|a(t) - p(t)|}{\log t} = +\infty.$$

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Example

- The fractional parts of the sequences n^c (for non-integer $c > 0$), $n \log n$ and $n^2\sqrt{2} + n\sqrt{3}$ are uniformly distributed in $[0, 1)$, while the fractional parts of the sequences $\log n$ and $n^2 + \sqrt{\log n}$ are not.

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- An important implication is that if the function $a \in \mathcal{H}$ stays "logarithmically" away from real multiples of integer polynomials, then the corresponding ergodic averages converge to the integral $\int f \ d\mu$ in ergodic systems.

The jointly ergodic setting

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T.-2021

Assume that the functions $a_1, \ldots, a_k \in \mathcal{H}$ have polynomial growth and that every non-trivial linear combination $a \in \mathcal{H}$ of them satisfies

$$\lim_{t\to+\infty} \frac{|a(t)-p(t)|}{\log t} = +\infty \text{ for every } \; p(t) \in \mathbb{Q}[t].$$

Then, for any ergodic measure preserving system (X, μ, T) and functions $f_1, \ldots, f_k \in L^\infty(\mu)$, we have

$$\frac{1}{N}\sum_{n=1}^{N} T^{\lfloor a_1(n)\rfloor} f_1 \ldots T^{\lfloor a_k(n)\rfloor} f_k \xrightarrow{L^2(\mu)} \int f_1 \, d\mu \cdots \int f_k \, d\mu.$$

Some previous results

Frantzikinakis-2010

The previous theorem holds in the case of distinct fractional powers, that is when $a_i(t) = t^{c_i}$ for pairwise distinct positive non-integers c_i. The fact that c_i is not an integer is necessary, otherwise there are ergodic, but not totally ergodic systems, where the limit is not the product of the integrals.

Karageorgos,Koutsogiannis-2017

The previous theorem holds in the case when the functions are non-integer polynomials that are linearly independent over $\mathbb{Q}$.

Some previous results

Bergelson,Moreira,Richter-2020

Let $\nabla - \text{span}\{a_1, ..., a_k\}$ denote the set of linear combinations of the functions $a_1, ..., a_k \in \mathcal{H}$ and their derivatives. Then, the previous theorem holds in the case that every function $f \in \nabla - \text{span}\{a_1, ..., a_k\}$ and every polynomial $p \in \mathbb{R}[t]$, we have

$$\lim_{t \to +\infty} \frac{|f(t) - p(t)|}{\log t} = +\infty.$$

A note on multiple recurrence

As a corollary of our main theorem, we have that, for any measure-preserving system (X, μ, T) and set A of positive measure, the following relation holds:

$$\lim_{n\to+\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-\lfloor a_1(n)\rfloor}A \cap \cdots \cap T^{-\lfloor a_k(n)\rfloor}A) \geq (\mu(A))^{k+1}.$$

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Corollary

If the set $\Lambda \subset \mathbb{N}$ has positive upper density, then

$$\liminf_{n \to +\infty} \frac{1}{N} \sum_{n=1}^{N} d^{*}\left(\Lambda \cap (\Lambda - \lfloor a_1(n) \rfloor) \cap \cdots \cap (\Lambda - \lfloor a_k(n) \rfloor)\right) \geq \left(d^{*}(\Lambda)\right)^{k+1}.$$

A similar result was obtained by Bergelson-Moreira-Richter with $\limsup$ in place of $\liminf$. They include a larger class of sequences, for which we have nice multiple recurrence results, but not a mean convergence result.

Joint ergodicity of sequences

In order to prove multiple convergence results, we use techniques to reduce the original convergence problem to the Host-Kra factors then, solve the problem in the case of nilsystems. Often, this second property is difficult to establish, but we can avoid it in the case where we study convergence to the product of the integrals.

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Definition

The sequences $a_1(n), \ldots, a_k(n)$ of integers are called jointly ergodic, iff for any ergodic system (X, μ, T) and functions $f_1, \ldots, f_k \in L^\infty(\mu)$ we have

$$\frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)} f_1 \ldots T^{a_k(n)} f_k \xrightarrow{L^2(\mu)} \int f_1 \, d\mu \cdots \int f_k \, d\mu.$$

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- A recent theorem of Frantzikinakis gives necessary and sufficient conditions for general sequences of integers to be jointly ergodic.

Joint ergodicity of sequences

Definition

i) The sequences $a_1(n), \ldots, a_k(n)$ of integers are called good for seminorm estimates if there exists $s \in \mathbb{N}$ such that for any system (X, μ, T) and functions $f_1, \ldots, f_k \in L^\infty(\mu)$ with $\| f_i \|_s = 0$ for some $i \in \{1, \ldots, k\}$, we have

$$\frac{1}{N} \sum_{n=1}^{N} T^{a_1(n)} f_1 \ldots T^{a_k(n)} f_k \xrightarrow{L^2(\mu)} 0.$$

ii) The sequences $a_1(n), \ldots, a_k(n)$ of integers are called good for equidistribution if for any $t_1, \ldots, t_k \in [0,1)$ (not all zero), we have

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} e(t_1 a_1(n) + \cdots + t_k a_k(n)) = 0.$$

Joint ergodicity of sequences

Frantzikinakis-2021

The following are equivalent:

a) The sequences $a_1(n), ..., a_k(n)$ are jointly ergodic.

b) The sequences $a_1(n), ..., a_k(n)$ are good for seminorm estimates and equidistribution.

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Frantzikinakis-2021

The following are equivalent:
a) The sequences $a_1(n), ..., a_k(n)$ are jointly ergodic.
b) The sequences $a_1(n), ..., a_k(n)$ are good for seminorm estimates and equidistribution.

- In our problem, we have to establish that the sequences $\lfloor a_1(n) \rfloor, ..., \lfloor a_k(n) \rfloor$ (for Hardy field functions $a_1, ..., a_k$) are good for seminorm estimates. The good for equidistribution property follows easily from our hypothesis on the linear combinations and the equidistribution results of Boshernitzan.

The weak-mixing case

We can actually get seminorm estimates under weaker assumptions.

T.-2021

Let $a_1, ..., a_k$ be functions in $\mathcal{H}$ such that $a_1, a_1 - a_2, ..., a_1 - a_k$ dominate $\log t$. Then, there exists a positive integer s such that for any measure preserving system (X, μ, T) and bounded functions $f_1, \ldots, f_k$ with $\|\!| f_1 |\!\|_s = 0$, we have

$$\lim_{N \to +\infty} \Big\| \frac{1}{N} \sum_{n=1}^{N} T^{\lfloor a_1(n) \rfloor} f_1 \ldots T^{\lfloor a_k(n) \rfloor} f_k \Big\|_{L^2(\mu)} = 0.$$

The weak-mixing case

In the case of weak mixing systems, we get the following theorem, which generalizes a result of Bergelson and Häland-Knutson.

T.-2021

Let $a_1, ..., a_k$ be functions in $\mathcal{H}$ such that $a_i, a_i - a_j$ dominate $\log t$ for all admissible values of i, j. Then, for any measure preserving system (X, μ, T) and bounded functions $f_1, \ldots, f_k$, we have

$$\lim_{N\to+\infty} \Big\|\frac{1}{N}\sum_{n=1}^{N} T^{\lfloor a_1(n)\rfloor} f_1 \ldots T^{\lfloor a_k(n)\rfloor} f_k - \int f_1 \, d\mu \cdots \int f_k \, d\mu\Big\|_{L^2(\mu)} = 0.$$

Sketch of the proof

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- We then apply the van der Corput inequality and use a PET induction scheme that reduces the complexity of the polynomials appearing in the averages.
- Due to some technical obstructions we have to use a double-averaging trick to separate slow-growing functions from faster growing functions.

An example

Consider the averages $\mathbb{E}_{n\in\mathbb{N}} T^{\lfloor n\log n\rfloor} f \cdot T^{\lfloor n^{2/3}\rfloor} g$. It suffices to show that

$$\mathop{\mathbb{E}}_{N\leq n\leq N+N^{0.51}} T^{\lfloor n\log n\rfloor} f \cdot T^{\lfloor n^{2/3}\rfloor} g$$

goes to 0, under the assumption that $|\!|\!| f |\!|\!|_s = 0$. However, we can approximate

$$(N+h)\log(N+h) = \frac{h^2}{2N} + \text{ smaller degree terms } + o_N(1),\ 0\leq h\leq N^{0.51}$$

and

$$(N+h)^{2/3} = \frac{2h}{3N^{1/3}} + \text{ smaller degree terms} + o_N(1),\ 0\leq h\leq N^{0.51}.$$

Thus, our problem reduces to showing that

$$\mathop{\mathbb{E}}_{0\leq h\leq N^{0.51}} T^{\lfloor \frac{h^2}{2N}+\dots\rfloor} f \cdot T^{\lfloor \frac{2h}{3N^{1/3}}+\dots\rfloor} g \to 0 \qquad (*)$$

and the iterates have polynomial form now.

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- In general, if a function grows faster than some fractional power t^δ, while the second function grows slower than all the fractional powers t^δ, we encounter the same problem.

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- In general, if a function grows faster than some fractional power t^δ, while the second function grows slower than all the fractional powers t^δ, we encounter the same problem.
- In order to combat this, we average again the averages over those small intervals. In (*), we have to average over $h \in [0, N^{0.51}]$ and over $N \in \mathbb{N}$.

Second example

Consider the averages $\mathop{\mathbb{E}}_{n\in\mathbb{N}} T^{\lfloor n\log n+\log^2 n\rfloor}f\cdot T^{\lfloor n\log n\rfloor}g$. Using an elementary lemma, our result follows if we show

$$\mathop{\mathbb{E}}_{1\le r\le R}\ \mathop{\mathbb{E}}_{0\le h\le r^{0.51}} T^{\lfloor (r+h)\log(r+h)+\log^2(r+h)\rfloor}f\cdot T^{\lfloor (r+h)\log(r+h)\rfloor}g\to 0.$$

We use the same Taylor expansion as before for the term $(r+h)\log(r+h)$. We can show that $\max_{0\le h\le r^{0.51}}|\log^2(r+h)-\log^2 r|=o_r(1)$. Combining the two, we want to bound the averages

$$\mathop{\mathbb{E}}_{1\le r\le R}\ \mathop{\mathbb{E}}_{0\le h\le r^{0.51}} T^{\lfloor \frac{h^2}{2r}+\dots\rfloor}\left(g\cdot T^{\lfloor \log^2 r\rfloor}f\right)=\mathop{\mathbb{E}}_{1\le r\le R}A_r, \tag{1}$$

We set $f_r=g\cdot T^{\lfloor \log^2 r\rfloor}f$ for convenience.

Second example

We want to bound the inner average for fixed (large) r. We write $h = k\lfloor\sqrt{2r}\rfloor + s$ for $0 \leq h \leq r^{0.51}$. After some simplifications, the inner average takes the form

$$\mathop{\mathbb{E}}_{0\leq k\leq Cr^{0.01}} T^{\lfloor k^2 + \text{smaller order monomials}\rfloor} f_r$$

We apply the van der Corput inequality twice to bound A_r:

$$\|A_r\|^4_{L^2(\mu)} \ll \frac{1}{M} + \mathop{\mathbb{E}}_{-M\leq m_1,m_2\leq M} \left| \int \bar{f}_r \cdot T^{2m_1m_2} f_r \right| + o_r(1) \text{ (for any } M > 0). \quad (2)$$

Comment on the PET bounds

More generally, we can show a bound that "looks like"

$$\Big\| \mathop{\mathbb{E}}_{n \le r} T^{\lfloor a_r k^2 \rfloor} f \Big\|_{L^2(m)}^4 \ll \frac{1}{M} + \mathop{\mathbb{E}}_{-M \le m_1, m_2 \le M} \Big| \int \bar{f} \cdot T^{\lfloor 2 m_1 m_2 a_r \rfloor} f \Big| + o_r(1)$$

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- The leading coefficients of the original polynomials and of their pairwise differences dictate the polynomial bounds (the polynomials on the shifts m_i) that we get above.

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$$\text{(for any } M > 0).$$

- The leading coefficients of the original polynomials and of their pairwise differences dictate the polynomial bounds (the polynomials on the shifts m_i) that we get above.
- In general, the original polynomials have coefficients that depend on r, so one needs to keep track of them until the end. They have a "specific" form, so they are removed using a special case of the main theorem, which is proven independently.

Putting the bound (2) in (1), we arrive at the expression

$$\mathop{\mathbb{E}}_{1\le r\le R}\ \mathop{\mathbb{E}}_{|m_1|,|m_2|\le M}\Big|\int \overline{(g\cdot T^{\lfloor \log^2 r\rfloor} f)}\cdot T^{2m_1m_2}(g\cdot T^{\lfloor \log^2 r\rfloor} f)\ d\mu\Big| =$$

$$\mathop{\mathbb{E}}_{|m_1|,|m_2|\le M}\ \mathop{\mathbb{E}}_{1\le r\le R}\Big|\int (\bar{g}\cdot T^{2m_1m_2} g)\cdot T^{\lfloor \log^2 r\rfloor}(\bar{f}\cdot T^{2m_1m_2} f)\ d\mu\Big|.$$

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In our example, we can simply use the $k=1$ case of the theorem, to deduce that the limit over R of the inner average is $\ll \|\!|\bar{f}\cdot T^{2m_1m_2} f|\!\|_2$. Thus, the original limit bounded (for any $M>0$) by

$$\mathop{\mathbb{E}}_{|m_1|,|m_2|\le M}\|\!|\bar{f}\cdot T^{2m_1m_2} f|\!\|_2.$$

We send $M\to+\infty$ and expand the seminorm inside by the definition, and we arrive at an iterated limit of polynomial averages. These are good for seminorm estimates (Leibman) and we are done.

The "linearly dependent" case

- When there are linear dependencies between the functions $a_1, \ldots, a_k$, then we can expect mean convergence, but not to the correct limit (i.e the product of the integrals). In this case, we cannot use the joint ergodicity characterization of Frantzikinakis.

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- The simplest example is when our functions have the form $a(t), b(t), a(t) + b(t)$ (e.g. $t \log t,\ t^{1.5},\ t^{1.5} + t \log t$).

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- In this case, we get the seminorm estimates as above and then we have to establish convergence results in nilmanifolds (namely, we will rely on the Host-Kra structure theorem)

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T.-2022

Let $\delta > 0$ and suppose $a_1, \dots, a_k \in \mathcal{H}$ have polynomial growth. Assume that every non-trivial linear combination a satisfies either

$$\lim_{t\to+\infty} \frac{|a(t)-p(t)|}{t^{\delta}} = +\infty \text{ for any polynomial } p(t) \in \mathbb{Q}[t]$$

or

$$\lim_{t\to+\infty} a(t) \text{ is a real number.}$$

Then, for any measure preserving system (X, μ, T) and any functions $f_1, \dots, f_k \in L^{\infty}(\mu)$, we have that the averages

$$\frac{1}{N}\sum_{n=1}^{N} T^{\lfloor a_1(n)\rfloor} f_1 \dots T^{\lfloor a_k(n)\rfloor} f_k$$

converge in $L^2(\mu)$.

The "linearly dependent" case

- The methods used to prove this theorem are similar to the ones above:
  i) Simultaneously approximate the original functions by polynomials in short intervals.
  ii) Apply the quantitative equidistribution results of Green-Tao for finite polynomial orbits on nilmanifolds.

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- This explains the appearance of the term t^{δ} in the statement of the previous theorem instead of the conjectured $\log t$.
- It is conjectured (Frantzikinakis) that we have mean convergence for the functions $a_1, \ldots, a_k$, if and only if, for all $t_1, \ldots, t_k \in [0, 1)$, the averages

$$\frac{1}{N}\sum_{n=1}^{N} e(t_1\lfloor a_1(n)\rfloor + \cdots + t_k\lfloor a_k(n)\rfloor)$$

  converge, i.e if we have mean convergence for rotations on the torus.

Thank You!