Distribution of translation orbits along good averaging sequences

speaker: Máté Wierdl

Nilpotent structures in topological dynamics, ergodic theory and combinatorics
Będlewo, June 9, 2023

Aleksander Rajchman

Plan

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1. Reminders
2. Definition of a good sequence (set)
3. Main question
   - Fourier coefficients of a measure
   - Appreciating the question
4. Results
   - Rational α
   - Irrational α
   - Example: how to have an interval to be the RN derivative
   - Non-Rajchman measures, $0-1$ law
   - Limit in Wiener–Wintner's theorem
5. Remaining questions
   - Single α
   - All α
6. Remarks on the proof

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Reminders

Mean ergodic theorem (MET) In a dynamical system (X, m, T) and $f \in L^2(X)$ we have $\lim_N \mathbb{A}_{n\in[0,N)} f \circ T^n = Pf$ where $\mathbb{A}_{n\in[0,N)} f_n := \frac{1}{N}\sum_{n\in[0,N)} f_n$, so $\mathbb{A}$ is the "average operator", and Pf is the T-invariant part of f.
By the *spectral theorem* and using Weyl's notation $\mathrm{e}(\theta) := e^{2\pi i\theta}$, MET is a consequence of $\lim_N \mathbb{A}_{n\in[0,N)} \mathrm{e}(n\alpha)$ existing for every real α and being equal 0 if $\alpha \notin \mathbb{Z}$ and 1 if $\alpha \in \mathbb{Z}$.

MET along squares says $\lim_N \mathbb{A}_{n\in[0,N)} f \circ T^{n^2}$ exists for $f \in L^2(X)$. By the spectral theorem, it's a consequence of $\lim_N \mathbb{A}_{n\in[0,N)} \mathrm{e}(n^2\alpha)$ existing for every α, which is true by Weyl's result. For irrational α the limit is 0 and for rational $\alpha = a/q$, $(a,q) = 1$, it's $\mathbb{A}_{r\in[0,q)} \mathrm{e}(r^2 a/q)$.

We have MET along many other sequences, such as primes, and the limit for irrational α is usually 0.

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Definition of a good sequence (set)

$$\mathbb{A}_S f = \mathbb{A}_{s\in S} f(s) := \frac{1}{\#S}\sum_{s\in S} f(s)$$
$$\mathrm{e}(\theta) := \mathrm{e}^{2\pi i\theta}$$

Definition

Let $s_1 < s_2 < \dots < s_n < \dots$ be a strictly increasing sequence of positive integers.
We say $S := (s_n)$ is good if for every real α the sequence $\left(\mathbb{A}_{n\in[1,N]}\,\mathrm{e}(s_n\alpha)\right)_N$ converges.

- Equivalently, the sequence $\left(\mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}\right)_N$ converges weakly. ($\mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ is a discrete probability measure on the torus $\mathbb{T}$.)
  We denote $\mu_{S,\alpha} := \lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$.
- Equivalently (by the spectral theorem), in every dynamical system (X, m, T) and $f \in L^2(X)$ the sequence $\left(\mathbb{A}_{n\in[1,N]} f(T^{s_n}x)\right)_N$ converges in L^2 norm.

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Main question

For a good set $S=(s_n)$, denote $\mu_{S,\alpha} := \lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$. Being the weak limit of Borel probability measures on $\mathbb{T}$, $\mu_{S,\alpha}$ is a Borel probability measure on $\mathbb{T}$.

Main Question

What can the limit measure $\mu_{S,\alpha}$ be?

- Can $\mu_{S,\alpha}$ be any Borel probability measure on $\mathbb{T}$?
- What $\mu_{S,\alpha}$ actually can be depends on α.
- The case of rational and irrational α are very different.
- Note that, by Weyl's result, the Haar-Lebesgue probability measure λ of those $\alpha\in\mathbb{T}$ for which $\mu_{S,\alpha}\neq\lambda$ is 0.

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- For a Borel measure ν on $\mathbb{T}$ and continuous $\mathbb{T} \to \mathbb{C}$ function ϕ, we use the functional notation $\nu(\phi) = \int_{\mathbb{T}} \phi \, \mathrm{d}\nu$. In particular, $\nu(\mathrm{e}^p)$ for $p \in \mathbb{Z}$ is the pth Fourier coefficient of ν.
- For the Haar-Lebesgue probability measure λ on $\mathbb{T}$ we have $\lambda(\mathrm{e}^p) = 0$ if $p \in \mathbb{Z}, p \neq 0$ and $\lambda(\mathrm{e}^0) = \lambda(1) = 1$.
- By Weierstrass' approximation theorem, $\mu_{S,\alpha} = \lim_N \mathbb{A}_{n\in[1,N]} \delta_{s_n\alpha}$ is a consequence of the existence of $\lim_N \mathbb{A}_{n\in[1,N]} \mathrm{e}^p(s_n\alpha)$ for every $p \in \mathbb{Z}$. Note that

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$\mu_{S,\alpha} := \lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$; equivalently $\mu_{S,\alpha}(e^p) = \lim_N \mathbb{A}_{n\in[1,N]} e^p(s_n\alpha)$ for every $p \in \mathbb{Z}$.

To appreciate the main question, note that for a given Borel probability measure ν on $\mathbb{T}$ and irrational α we can construct a set $S \subset \mathbb{N}$ so that $\mu_{S,\alpha} = \nu$.
This can be done in multiple ways. In case of a point mass, say, at $1/2$, the construction is particularly simple since all we need is to have $\lim_n s_n\alpha = 1/2 \mod 1$ which can be aciheved since $\{ n\alpha \mid n \in \mathbb{N} \}$ is dense $\mod 1$.

Definition (Representation of a measure)

Given a real number α and a Borel probability measure ν, we say ν is representable at α if there is a good set S with $\mu_{S,\alpha} = \nu$.

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Results

Rational α

α is rational, $\alpha = \frac{a}{q}$, $(a, q) = 1$.

$\mu_{S,a/q} := \lim_N \mathbb{A}_{n\in[1,N]} \delta_{s_n a/q}$, $\mu_{S,a/q}(\mathrm{e}^p) = \lim_N \mathbb{A}_{n\in[1,N]} \mathrm{e}\left(s_n p \frac{a}{q}\right)$.

$\mu_{S,a/q}$ is supported on the set $\mathbb{T}_q = \left\{ \frac{b}{q} \,\middle|\, 0 \le b < q \right\}$ of qth roots of unity.

For example, by the prime number theorem in arithmetic progressions, if s_n is the nth prime number then $\mu_{S,a/q}$ is the uniform probability measure on $\left\{ \frac{b}{q} \,\middle|\, 0 \le b < q, (b, q) = 1 \right\} \subset \mathbb{T}_q$, so $\mu_{S,a/q}\{b/q\} = \frac{1}{\phi(q)}$ for every b where ϕ is Euler's totient function.

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Let ν be any probability measure on $\mathbb{T}_q$ and $\frac{a}{q} \in \mathbb{T}_q$.
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Example: how to have an interval to be the RN derivative

Let $I \subset \mathbb{T}$ be an interval, and let $\nu = \frac{1}{\lambda(I)} \mathbb{1}_I \cdot \lambda$.

For a given irrational α, Let us define the set $S \subset \mathbb{N}$ by

$$S := \{ n \mid n \in \mathbb{N}, n\alpha \in I \mod 1 \}$$

So if $I = (A, B)$ and $\alpha = \sqrt{2} - 1$, then

Writing S as a sequence $s_1 < s_2 < \dots$, $\mu_{S,\beta} = \lim_N \mathbb{A}_{n \in [1,N]} \delta_{s_n \beta}$ exists for every β and $\mu_{S,\alpha} = \frac{1}{\lambda(I)} \mathbb{1}_I \cdot \lambda$ as can be seen by using Weyl's uniform distribution theorem.

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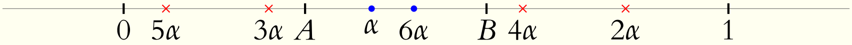

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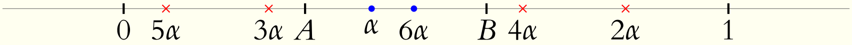

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Results
Irrational α

Let α be irrational.

Theorem (Lesigne-Quas-Rosenblatt-Wierdl)

For every good sequence $S = (s_n)$ the limit Borel probability measure $\mu_{S,\alpha}$ is continuous.

As a consequence, by Wiener's theorem, $\lim_P \mathbb{A}_{p \in [-P,P]} |\mu_{S,\alpha}(e^p)| = 0$.

Theorem (Lesigne-Wierdl)

If the Borel probability measure ν is Rajchman, that is, $\lim_{|p| \to \infty} \nu(e^p) = 0$, then ν can be represented at α.

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Main question
Non-Rajchman measures, $0-1$ law

Theorem (Lesigne-Quas-Rosenblatt-Wierdl)

If the Borel probability measure ν is non-Rajchman then there is a set $A \subset \mathbb{T}$ with $\lambda(A) = 1$ so that ν cannot be represented at any $\alpha \in A$.

We have a "$0-1$" law: Rajchman probability measures can be represented at every irrational α, so on a set A with $\lambda(A) = 1$, while non-Rajchman probability measures can be represented on a set A with $\lambda(A) = 0$ ($A = \emptyset$ might be possible, who knows?!).

Theorem (Cuny-Parreau)

There is a good set S and an irrational α so that $\mu_{S,\alpha}$ is non-Rajchman.

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Interlude.

Theorem (Wiener-Wintner)

Let $(X, \mathfrak{m}, T)$ be a dynamical system, and E a measurable set. Then $\lim_N \mathbb{A}_{n\in[0,N)} \mathbb{1}_E(T^n x)\,\mathrm{e}(n\beta)$ exists for a.e. x and every β.

So if T is ergodic and $\mathfrak{m}(E) > 0$ then for a.e. x the set $S := \{\, n \mid T^n x \in E \,\}$ is a good set.

What is $\mu_{S,\beta}$?

Well, if T is ergodic and $\mathfrak{m}(E) > 0$ then $S = S(x)$ has positive density for a.e. x: $\mathrm{d}(S) := \lim_N \mathbb{A}_{n\in[0,N)} \mathbb{1}_S(n) = \mathfrak{m}(E)$. It's not difficult to see that then $\mu_{S,\beta}$ is absolutely continuous with respect to λ for every β, so, in particular, it's Rajchman.

Results

Limit in Wiener-Wintner's theorem

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Limit in Wiener-Wintner's theorem

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Theorem (Wiener-Wintner)

Let $(X, \mathfrak{m}, T)$ be a dynamical system, and E a measurable set.
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Remaining questions

Single α

Remaining Question (For single α)

Which continuous Borel probability measures can be represented at some irrational α?

- If the continuous Borel probability measure ν is invariant with respect to multiplication by both 2 and 3 and $\nu \neq \lambda$ then, it cannot be represented at an irrational α.
  So if every continuous Borel probability measure can be represented at an irrational α, then a continuous, "$\times 2 \times 3$" invariant measure is the Haar-Lebesgue measure, and hence Furstenberg's conjecture holds.
- There is no [illegible] α where potentially every continuous measure can be represented: for every irrational α there is a continuous measure which cannot be represented at α.

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Given a collection $\mathcal{A} \subset \mathbb{T}$ of α's and a good sequence S, what can be the corresponding collection of limit measures $\{\, \mu_{S,\alpha} \mid \alpha \in \mathcal{A} \,\}$?

By the easier direction of Lyons characterization of Rajchman measures (already observed by Rajchman), for every good set S and Rajchman measure ν we have $\nu\{\, \alpha \mid \mu_{S,\alpha} \neq \lambda \,\} = 0$. In particular, since λ is Rajchman, $\lambda\{\, \alpha \mid \mu_{S,\alpha} \neq \lambda \,\} = 0$ (Weyl's result).

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Remarks on the proof

Definition (Weight, weighted average)

- We say a sequence $w = (w(n))_{n \in \mathbb{N}}$ is a weight if $w(n) \geq 0$ for every $n \in \mathbb{N}$.
- For a given non-identically 0 weight w, the w-weighted average $\mathbb{A}^w_{n \in [1,N]} f_n$ of the sequence $(f_n)_{n \in \mathbb{N}}$ is defined by

$$\mathbb{A}^w_{n \in [1,N]} f_n := \frac{1}{\sum_{n \in [1,N]} w(n)} \sum_{n \in [1,N]} w(n) f_n$$

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Theorem (Lesigne-Wierdl)

Let α be irrational.
If the Borel probability measure ν is Rajchman, that is, $\lim_{|p|\to\infty} \nu(\mathrm{e}^p) = 0$, then ν can be represented at α, so there is a good set S with $\mu_{S,\alpha} = \nu$.

- For a given Rajchman ν, we first construct a w so that the w-weighted averages converge to ν: We use Fejér's kernel for the Fourier series of ν to get the weight w so that $\lim_N \mathbb{A}^w_{n\in[1,N]}\delta_{n\beta}$ exists for every β and $\lim_N \mathbb{A}^w_{n\in[1,N]}\delta_{n\alpha} = \nu$.
- We then randomly "construct" a good S which represents ν at α, that is, $\mu_{S,\alpha} = \nu$, using w as "expectation". We show that almost every set S satisfies $\lim_N \left(\mathbb{A}_{s\in S(N)}\delta_{s\alpha} - \mathbb{A}^w_{n\in[1,N]}\delta_{n\alpha} \right) = 0$, where $S(N) = S \cap [1,N]$.
- The good set S representing ν can be taken to be a subset of your favorite set: the set of squares, primes, or randomly generated set.

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Theorem

Let α be irrational, and ν be a continuous Borel probability measure which is invariant with respect to multiplication by 2 and 3. Suppose that for a good set S we have $\mu_{S,\alpha} = \nu$. Then $\nu = \lambda$.

- Suppose, indirecte, that $\nu \neq \lambda$. Then there is $p \in \mathbb{Z}$, $p \neq 0$, so that $\nu(\mathrm{e}^p) \neq 0$.
- Since ν is assumed to be $\times 2 \times 3$-invariant, for every j, k we have $\nu(\mathrm{e}^{p2^j3^k}) = \nu(\mathrm{e}^p)$. But we have $\mu_{S,p2^j3^k\alpha}(\mathrm{e}) = \mu_{S,\alpha}(\mathrm{e}^{p2^j3^k}) = \nu(\mathrm{e}^p)$.
- By Furstenberg's theorem, the set $\left\{ p2^j3^k\alpha \,\middle|\, j,k \in \mathbb{Z} \right\}$ is dense in $\mathbb{T}$, so $|\mu_{S,\beta}(\mathrm{e})| = |\nu(\mathrm{e}^p)| > 0$ for a dense set of β, namely for β of the form $\beta = p2^j3^k\alpha$. But $\mu_{S,\beta}(\mathrm{e}) = 0$ for a dense set of β as well (the set of such β is of λ-measure 1 by Weyl's theorem). Since the function $\phi(\beta) := |\mu_{S,\beta}(\mathrm{e})|$ is a limit of continuous functions, this is impossible (by Baire's theorem).

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