Distribution of translation orbits along good averaging sequences speaker: Máté Wierdl

Nilpotent structures in topological dynamics, ergodic theory and combinatorics Będlewo, June 9, 2023

Aleksander Rajchman



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- Main question Fourier coefficients Appreciation
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MET along squares says lim_N A_{n∈[0,N}∫ ◦ T^{n²} exists for f ∈ L²(X). By the spectral theorem, it's a consequence of lim_N A_{n∈[0,N}) e(n²α) existing for every α, which is true by Weyl's result. For irrational α the limit is 0 and for rational α = a/q, (a, q) = 1, it's A_{n∈[0,q}) e(r²a/q).

We have MET along many other sequences, such as primes, and the limit for irrational *α* is usually 0.

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Definition

Let $s_1 < s_2 < \cdots < s_n < \ldots$ be a strictly increasing sequence of positive integers. We say $S := (s_n)$ is good if for every real α the sequence $\left(\mathbb{A}_{n \in [1,N]} e(s_n \alpha)\right)_N$ converges.

■ Equivalently, the sequence $\left(\mathbb{A}_{n\in[1,N]}\delta_{s,a}\right)_N$ converges weakly. $\left(\mathbb{A}_{n\in[1,N]}\delta_{s,a}\right)_{s,a}$ is a discrete probability measure on the torus T.) We denote $\mu_{S,a} := \lim_N \mathbb{A}_{n\in[1,N]}\delta_{s,a}$.

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$$\begin{array}{l} \mathbb{A}_{S}f = \mathbb{A}_{s \in S}f(s) \coloneqq \frac{1}{\#S}\sum_{s \in S}f(s) \\ \mathbf{e}(\theta) \coloneqq \mathbf{e}^{2\pi i\theta} \end{array}$$

Definition

Let $s_1 < s_2 < \cdots < s_n < \ldots$ be a strictly increasing sequence of positive integers. We say $S := (s_n)$ is good if for every real α the sequence $(\mathbb{A}_{n \in [1,N]} \mathbf{e}(s_n \alpha))_N$ converges.

 Equivalently, the sequence (A_{n∈[1,N]}δ_{s_nα})_N converges weakly. (A_{n∈[1,N]}δ_{s_nα} is a discrete probability measure on the torus T.) We denote μ_{S,α} := lim_N A_{n∈[1,N]}δ_{s_nα}.

• Equivalently (by the spectral theorem), in every dynamical system (X, \mathfrak{m}, T) and $f \in L^2(X)$ the sequence $\left(\mathbb{A}_{n \in [1,N]} f(T^{s_n} x)\right)_{x_1}$ converges in L^2 -norm.

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 $\begin{aligned} \mathbf{A}_{\mathcal{S}} f &= \mathbf{A}_{s \in \mathcal{S}} f(s) \coloneqq \frac{1}{\# S} \sum_{s \in \mathcal{S}} f(s) \\ \mathbf{e}(\theta) &\coloneqq \mathbf{e}^{2\pi i \theta} \end{aligned}$

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Let $s_1 < s_2 < \cdots < s_n < \ldots$ be a strictly increasing sequence of positive integers. We say $S := (s_n)$ is good if for every real α the sequence $\left(\mathbb{A}_{n \in [1,N]} e(s_n \alpha)\right)_N$ converges.

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Questions Single α All α Proof $\times 2 \times 3$ For a good set $S = (s_n)$, denote $\mu_{S,\alpha} := \lim_{N \to \infty} \mathbb{A}_{n \in [1,N]} \delta_{s_n \alpha}$. Being the weak limit of Borel probability measures on \mathbb{T} , $\mu_{S,\alpha}$ is a Borel probability measure on \mathbb{T} .

Main Question

- Can $\mu_{S,\alpha}$ be any Borel probability measure on T?
- What $\mu_{S,\alpha}$ actually can be depends on α .
- The case of rational and irrational α are very different
- Note that, by Weyl's result, the Haar–Lebesgue probability measure λ of those $\alpha \in \mathbb{T}$ for which $\mu_{S,\alpha} \neq \lambda$ is 0.

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- For a Borel measure ν on \mathbb{T} and continuous $\mathbb{T} \to \mathbb{C}$ function ϕ , we use the functional notation $\nu(\phi) = \int_{\mathbb{T}} \phi \, d\nu$. In particular, $\nu(e^p)$ for $p \in \mathbb{Z}$ is the *p*th Fourier coefficient of ν .
- For the Haar-Lebesgue probability measure λ on \mathbb{T} we have $\lambda(e^p) = 0$ if $p \in \mathbb{Z}, p \neq 0$ and $\lambda(e^0) = \lambda(1) = 1$.
 - By Weierstrass' approximation theorem, μ_{S,e} = lim_N A_{n∈[1,N]}δ_{s,e} is a consequence of the existence of lim_N A_{n∈[1,N]} e^p(s_nα) for every $p \in \mathbb{Z}$. Note that
 - $\lim_{N \to \infty} \Lambda_{n \in [1,N]} \, \mathbf{e}^{p}(s_{\theta} \alpha) = \lim_{N} \Lambda_{n \in [1,N]} \, \mathbf{e}(s_{\eta} \alpha) = \mu_{S, \mu_{n}}(\mathbf{e}) \, .$

- For a Borel measure ν on \mathbb{T} and continuous $\mathbb{T} \to \mathbb{C}$ function ϕ , we use the functional notation $\nu(\phi) = \int_{\mathbb{T}} \phi \, d\nu$. In particular, $\nu(e^p)$ for $p \in \mathbb{Z}$ is the *p*th Fourier coefficient of ν .
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Fourier coefficients

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To appreciate the main question, note that for a given Borel probability measure ν on \mathbb{T} and irrational α we can construct a set $S \subset \mathbb{N}$ so that $\mu_{S,\alpha} = \nu$. This can be done in multiple ways. In case of a point mass, say, at 1/2, the construction is particularly simple since all we need is to have $\lim_{n \to \infty} s_n \alpha = 1/2$ mod 1 which can be achieved since $\{n\alpha \mid n \in \mathbb{N}\}$ is dense mod 1.

Definition (Representation of a measure)

Given a real number α and a Borel probability measure ν , we say ν is representable at α if there is a good set *S* with $\mu_{S,\alpha} = \nu$.

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Questions Single α All α Proof $\times 2 \times 3$ α is rational, $\alpha = \frac{a}{q}$, (a, q) = 1.

 $\mu_{S,a/q} \coloneqq \lim_{N \to \infty} \mathbb{A}_{n \in [1,N]} \delta_{s_n a/q}, \, \mu_{S,a/q}(\mathbf{e}^p) = \lim_{N \to \infty} \mathbb{A}_{n \in [1,N]} \, \mathbf{e}\Big(s_n p_{\frac{q}{q}}\Big).$

 $\mu_{S,a/q}$ is supported on the set $\mathbb{T}_q = \left\{ \left. \frac{b}{q} \right| 0 \le b < q \right\}$ of qth roots of unity.

For example, by the prime number theorem in arithmetic progressions, if s_n is the th prime number then $\mu_{S,a/q}$ is the uniform probability measure on

 $\left| b \leq b < q, (b,q) = 1 \right| \subset \mathbb{T}_q$, so $\mu_{S,a/q}\{b/q\} = \frac{1}{\phi(q)}$ for every b where ϕ is a ler's totient function.

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Theorem (Lesigne-Quas-Rosenblatt-Wierdl)

Let v be any probability measure on \mathbb{T}_q and $\frac{a}{q} \in \mathbb{T}_q$. Then v is representable at $\frac{a}{q}$, that is, there is a good set S so that $\mu_{S,a/q} = v$.

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If $s_n = n$, n^2 , $\lfloor n^3 \log n \rfloor$ or the *n*th prime number then $\mu_{S,\alpha} = \lambda$, since $\mu_{S,\alpha}(\mathbf{e}^p) = 0$ for every nonzero $p \in \mathbb{Z}$.

San we get the limit measure to be something other than λ ?

Let us try absolutely continuous measures.

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- Let us try absolutely continuous measures.

Let $I \subset \mathbb{T}$ be an interval, and let $\nu = \frac{1}{\lambda(I)} \mathbb{1}_I \cdot \lambda$. For a given irrational α , Let us define the set $S \subset \mathbb{N}$ by

$$S := \{ n \mid n \in \mathbb{N}, n\alpha \in I \mod 1 \}$$

So if I = (A, B) and $\alpha = \sqrt{2} - 1$, then

$$0 5\alpha 3\alpha A \alpha 6\alpha B 4\alpha 2\alpha 1$$

Writing S as a sequence $s_1 < s_2 < \dots, \mu_{S,\beta} = \lim_N \mathbb{A}_{n \in [1,N]} \delta_{s_n\beta}$ exists for every β and $\mu_{S,\alpha} = \frac{1}{\lambda(I)} \mathbb{1}_I \cdot \lambda$ as can be seen by using Weyl's uniform distribution theorem

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Let α be irrational.

Theorem (Lesigne-Quas-Rosenblatt-Wierdl)

For every good sequence $S = (s_n)$ the limit Borel probability measure $\mu_{S,\alpha}$ is continuous.

As a consequence, by Wiener's theorem, $\lim_{P \in [-P,P]} |\mu_{S,\alpha}(e^p)| = 0$.

Theorem (Lesigne-Wierdl

If the Borel probability measure v is Rajchman, that is, $\lim_{|p|\to\infty} v(e^p) = 0$, then v can be represented at α .

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Let α be irrational.

Theorem (Lesigne-Quas-Rosenblatt-Wierdl)

For every good sequence $S = (s_n)$ the limit Borel probability measure $\mu_{S,\alpha}$ is continuous.

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Theorem (Cuny-Parreau

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Let (X, \mathfrak{m}, T) be a dynamical system, and E a measurable set. Then $\lim_N \mathbb{A}_{n \in [0,N]} \mathbb{1}_E(T^n x) e(n\beta)$ exists for a.e. x and every β .

So if T is ergodic and $\mathfrak{m}(E) > 0$ then for a.e. x the set $S := \{ n \mid T^n x \in E \}$ is a good set.

What is $\mu_{S,\beta}$?

Well, if T is ergodic and $\mathfrak{m}(E) > 0$ then S = S(x) has positive density for a.e. x: $d(S) := \lim_N \mathbb{A}_{n \in [0,N)} \mathbb{1}_S(n) = \mathfrak{m}(E)$. It's not difficult to see that then $\mu_{S,\beta}$ is absolutely continuous with respect to λ for every β , so, in particular, it's Rajchman.

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Remaining Question (For single α)

Which continuous Borel probability measures can be represented at some irrational α?

- If the continuous Borel probability measure ν is invariant with respect to multiplication by both 2 and 3 and ν ≠ λ then, it cannot be represented at an irrational α.
 - So if every continuous Borel probability measure can be represented at an irrational α , then a continuous, " $\times 2 \times 3$ " invariant measure is the
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- There is no magical α where potentially every continuous measure can be represented: for every irrational α there is a continuous measure which cannot be represented at α.

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Remaining Question (All α)

Given a collection $\mathcal{A} \subset \mathbb{T}$ of α 's and a good sequence S, what can be the corresponding collection of limit measures { $\mu_{S,\alpha} \mid \alpha \in \mathcal{A}$ }?

By the easier direction of Lyons characterization of Rajchman measures (already observed by Rajchman), for every good set *S* and Rajchman measure ν we have ν { $\alpha \mid \mu_{S,\alpha} \neq \lambda$ } = 0. In particular, since λ is Rajchman, λ { $\alpha \mid \mu_{S,\alpha} \neq \lambda$ } = 0 (Weyl's result).

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Definition (Weight, weighted average)

- We say a sequence $w = (w(n))_{n \in \mathbb{N}}$ is a weight if $w(n) \ge 0$ for every $n \in \mathbb{N}$.
- For a given non-identically 0 weight w, the w-weighted average A^w_{n∈[1,N]}f_n of the sequence (f_n)_{n∈ℕ} is defined by

$$\mathbb{A}_{n\in[1,N]}^{w}f_{n} \coloneqq \frac{1}{\sum_{n\in[1,N]}w(n)}\sum_{n\in[1,N]}w(n)f_{n}$$

Theorem (Lesigne-Wierdl)

Let α be irrational.

If the Borel probability measure ν is Rajchman, that is, $\lim_{|p|\to\infty} \nu(e^p) = 0$, then ν can be represented at α , so there is a good set S with $\mu_{S,\alpha} = \nu$.

For a given Rajchman ν, we first construct a w so that the w-weighted averages converge to ν: We use Fejér's kernel for the Fourier series of ν to get the weight w so that lim_N A^w_{n∈[1,N]}δ_{nβ} exists for every β and lim_N A^w_{n∈[1,N]}δ_{nα} = ν.
We then randomly "construct" a good S which represents ν at α, that is, μ_{S,α} = ν, using w as "expectation". We show that almost every set S satisfies lim_N (A_{s∈S(N)}δ_{sα} - A^w_{n∈[1,N]}δ_{nα}) = 0, where S(N) = S ∩ [1, N].
The good set S representing ν can be taken to be a subset of your favorite set: the set of squares, primes, or randomly generated set.

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- Suppose, indirecte, that ν ≠ λ. Then there is p ∈ Z, p ≠ 0, so that ν(e^p) ≠ 0.
 Since ν is assumed to be ×2 × 3-invariant, for every j, k we have ν(e^{p2/3k}) = ν(e^p). But we have μ_{s,p2/2k}(e) = μ_{s,x}(e^{p2/3k}) = ν(e^p).
- By Furstenberg's theorem, the set $\left\{ p2^{j}3^{k}\alpha \mid j,k \in \mathbb{Z} \right\}$ is dense in \mathbb{T} , so $|\mu_{S,\beta}(\mathbf{e})| = |\nu(\mathbf{e}^{p})| > 0$ for a dense set of β , namely for β of the form $\beta = p2^{j}3^{k}\alpha$. But $\mu_{S,\beta}(\mathbf{e}) = 0$ for a dense set of β as well (the set of such β is of λ -measure 1 by Weyl's theorem). Since the function $\phi(\beta) := |\mu_{S,\beta}(\mathbf{e})|$ is a limit of continuous functions, this is impossible (by Baire's theorem).

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