Minicourse on nilspaces - part I

Introduction to nilspaces and connections to Gowers norms

Nilpotent Structures in Topological Dynamics,
Ergodic Theory and Combinatorics

Będlewo, Poland, June 5-th, 2023

Origins in combinatorial number theory and ergodic theory

Szemerédi's theorem: for every $k \in \mathbb{N}$, every subset of $\mathbb{Z}$ of positive upper density contains arithmetic progressions of length k.

- Szemerédi proved this in 1975 using mainly combinatorics and graph theory.
- Furstenberg gave a new proof in 1977 using ergodic theory.
- In 1953 Roth had given an effective proof for $k = 3$ using the circle method.

Fundamental idea: given a function f on a finite abelian group Z (or a finite interval in $\mathbb{Z}$), the Fourier transform of f enables a useful analysis of multilinear averages of f over patterns such as 3-APs (or solutions to other single linear equations), e.g. $\mathbb{E}_{x,r\in \mathrm{Z}} f(x)\, f(x+r)\, f(x+2r)$. In ergodic theory, a related idea is that the Kronecker factor is characteristic for ergodic averages over such patterns.

- Since the end of the 1990s, there has been a considerable **extension of this idea**, driven by the notion of **uniformity norms** introduced by Gowers in arithmetic combinatorics (1998), and by the analogous notion of **uniformity seminorms** introduced by Host and Kra in ergodic theory (2005).
- A key objective in this development has been to understand the relation between uniformity norms and certain **nilpotent structures**, especially via the study of the **cube structures** underlying these norms.

Uniformity norms and cubes in abelian groups

In his proof of Szemerédi's theorem, Gowers introduced the following norms.

Uniformity norms. Let G be a compact abelian group, let $d \geq 2$, and $f \in L^\infty(G)$. The Gowers U^d-norm of f consists of an integral over *standard* (or *degree 1*) **cubes of dimension** d in G (relative to Haar measure):

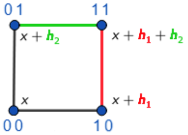
$$\|f\|_{U^d} = \Big(\int_{\text{cubes } c:\{0,1\}^d \to G} \big[\prod_{\text{vertices } v \in \{0,1\}^d} \mathcal{C}^{|v|} f(c(v)) \big] \, \mathrm{d}\mu_d(c) \Big)^{2^{-d}},$$

where $\mathcal{C}$ is the complex-conjugation operator ($\mathcal{C}f(x) = \overline{f(x)}$), and $|v| = \sum_{i=1}^d v_i$.

$$\|f\|_{U^2} = \big(\int_{x,h_1,h_2 \in G} f(x)\, \overline{f(x+h_1)}\, \overline{f(x+h_2)}\, f(x+h_1+h_2) \big)^{\frac{1}{4}}.$$

$\|\cdot\|_{U^2} \;\leftrightarrow\;$ 2-cubes $(x,\ x+h_1,\ x+h_2,\ x+h_1+h_2) \in G^4$,

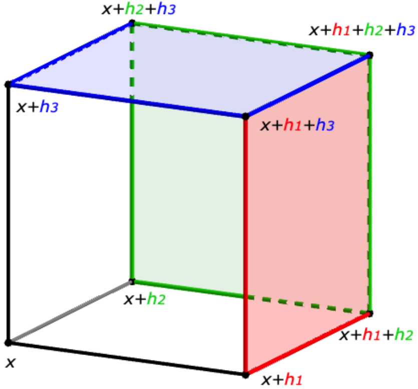
i.e. elements $c = (x + v_1 h_1 + v_2 h_2)_{v \in \{0,1\}^2} \in G^{\{0,1\}^2}$.



Uniformity norms and cubes in abelian groups

$$\| \cdot \|_{U^d} \;\leftrightarrow\; d\text{-cubes } c = (x + v_1 h_1 + \cdots + v_d h_d)_{v \in \{0,1\}^d} \in G^{\{0,1\}^d}.$$

E.g. for $d = 3$:



From uniformity norms to higher-order Fourier analysis

$$\|\cdot\|_{U^2} \;\leftrightarrow\; \text{Classical Fourier analysis:} \qquad \|f\|_{U^2} = \|\widehat{f}\|_{\ell^4}.$$

$\Rightarrow$ **Inverse theorem for** $\|\cdot\|_{U^2}$. If $f : G \to \mathbb{C}$ is 1-bounded and $\|f\|_{U^2} \geq \epsilon > 0$, then there is some character $\chi \in \widehat{G}$ such that $|\int_G f\,\overline{\chi}\,\mathrm{d}\mu| \geq \epsilon^2$.

Fundamental question in higher-order Fourier analysis

Which functions play the role of characters for the U^d-norm, for $d > 2$?

Some generalizations of Fourier characters

Fourier characters are defined using the **circle**: $x \mapsto e(\phi(x))$, $\phi \in \hom(G \to \mathbb{R}/\mathbb{Z})$.

The first higher-order generalizations of characters focused on two settings.

$\underline{G = \mathbb{Z}_N}$: inspired by ergodic theory, Green and Tao started using $(d-1)$-**step nilsequences**. These generalize characters by replacing $\mathbb{R}/\mathbb{Z}$ with a $(d-1)$-**step nilmanifold** H/Γ (quotient of a $(d-1)$-step nilpotent Lie group H by a lattice Γ).

E.g. **1-step**: $\mathbb{R}/\mathbb{Z} \quad \leftrightarrow$ character $\chi : n \mapsto e(\frac{t}{N} n)$

2-step: $\mathcal{H}/\Gamma = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} / \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \quad \leftrightarrow$ nilsequence $\chi : n \mapsto F(g^n\Gamma)$, $(g \in \mathcal{H})$.

Inverse Theorem for the U^d-norm on $\mathbb{Z}_N$ (Green, Tao, Ziegler, 2010)
Let $f : \mathbb{Z}_N \to \mathbb{C}$, $|f| \leq 1$, $\|f\|_{U^d} \geq \epsilon > 0$. Then there exists a $(d-1)$-step nilsequence $\chi : \mathbb{Z}_N \to \mathbb{C}$, of complexity $\ll_\epsilon 1$, such that $|\mathbb{E}_{x \in \mathbb{Z}_N} f(x)\overline{\chi(x)}| \gg_\epsilon 1$.

$\underline{G = \mathbb{F}_p^n}$: the stronger algebraic structure in this setting enabled a generalization of characters using **polynomial maps** $\mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$, with corresponding inverse theorems (Bergelson–Tao–Ziegler (high characteristic case), 2010; Tao–Ziegler, 2011).

Inverse Theorem for the U^d-norm on $\mathbb{F}_p^n$
Let $f : \mathbb{F}_p^n \to \mathbb{C}$, $|f| \leq 1$, $\|f\|_{U^d} \geq \epsilon$. Then there exists a polynomial map $P : \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ of degree at most $d-1$ such that $|\mathbb{E}_{x \in \mathbb{F}_p^n} f(x)\overline{e(P(x))}| \gg_\epsilon 1$.

Generalizing cubes to understand uniformity norms

The previous inverse theorems confirmed and strengthened the connection between Gowers norms and nilpotent groups. They also led to new important questions.

Green (ICM 2014): "*For me the key open question is to find the "right" proof of the inverse conjecture for the Gowers norms. At the moment the proofs are unsatisfactory on a conceptual level*."

In 2005 Host and Kra had introduced **uniformity seminorms** in ergodic theory, crucial for their Ergodic Structure Theorem (a highlight in these ergodic theory developments, building on work of Furstenberg–Weiss, Conze–Lesigne and others).

Host and Kra later introduced **parallelepiped structures** (2006), initiating an axiomatic approach to the study of cube structures underlying uniformity seminorms (structures further developed in dynamics by Host–Kra–Maass in particular), toward a deeper understanding of the relation between these seminorms and nilpotent groups.

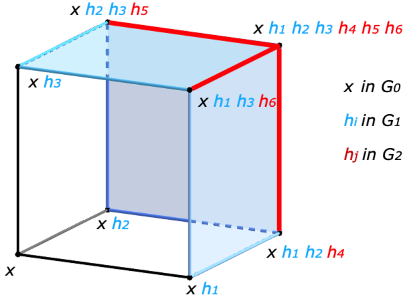
Such structures included cubes definable on any filtered nilpotent group $(G, G_\bullet)$, now known as **Host-Kra cubes**.

Generalizing cubes to understand uniformity norms

Recall that given a group G, a sequence of subgroups $G_\bullet = (G_i)_{i\geq 0}$ is a *filtration of degree k* on G if $G_0 = G_1 = G$ and $\forall\, i,j \geq 0$, $[G_i, G_j] \leq G_{i+j}$, and $G_{k+1} = \{\mathrm{id}\}$.

For each n the group of **Host-Kra cubes** of dimension n is the subgroup $C^n(G_\bullet)$ of $G^{\{0,1\}^n}$ consisting of elements (i.e. maps $c : \{0,1\}^n \to G$) of the following kind:

E.g. for $G_\bullet = (G_0, G_1, G_2)$ of degree 2 and $n = 3$:



Toward nilspaces via higher-order Fourier analysis

In 2009, Szegedy initiated an approach to higher-order Fourier analysis using in particular ultraproducts and ideas originating in graph-limit theory. As part of this approach, in 2010, Antolín-Camarena and Szegedy (inspired by the work of Host and Kra) introduced a general concept of spaces equipped with cube structures (which included in particular the Host-Kra cube structures), called **nilspaces**.

Nilspaces form a category that generalizes the category of compact abelian groups, and which includes nilmanifolds.

Nilspace theory yields a useful answer to the fundamental question of higher-order Fourier analysis, providing natural analogues of characters for general compact abelian groups.

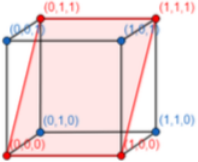
This theory is growing rapidly, with contributions by many authors: Antolín-Camarena, C., González-Sánchez, Gutman, Jamneshan, Manners, Shalom, Szegedy, Tao, Varjú, etc.

This theory relies on an abstract and general notion of **cube structures**.
To define this we need to start with the category of discrete cubes.

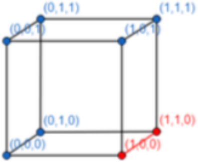
The category of discrete cubes

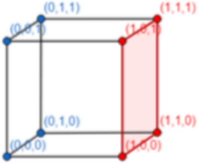
- For every $n \geq 0$, the **discrete n-cube** is $\{0,1\}^n$, now often denoted by $[\![n]\!]$.
- A **morphism** (of discrete cubes) is a function $\phi : [\![m]\!] \to [\![n]\!]$ which can be extended to an affine homomorphism $\mathbb{Z}^m \to \mathbb{Z}^n$.

Example of morphism: $\phi : [\![2]\!] \to [\![3]\!]$, $(v_1, v_2) \mapsto (v_1, v_2, v_2)$



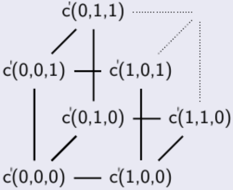
- For $0 \leq m \leq n$, an **m-face** of $[\![n]\!]$ is a set $F \subset [\![n]\!]$ which is determined by fixing $n - m$ coordinates of the vertex $v \in [\![n]\!]$.





Definition (Antolín-Camarena, Szegedy 2010). A **nilspace** is a set X equipped with a *cube set* $C^n(X) \subset X^{[\![n]\!]}$ for each $n \in \mathbb{Z}_{\geq 0}$ (the elements of $C^n(X)$ are the n-**cubes** on X), satisfying the following axioms:

1. (Composition) For every $c \in C^n(X)$ and every morphism $\phi : [\![m]\!] \to [\![n]\!]$, we have $c \circ \phi \in C^m(X)$.
2. (Ergodicity) $C^1(X) = X^{\{0,1\}}$.
3. (Corner completion) Let $c' : \{0,1\}^n \setminus \{1^n\} \to X$ be an n-**corner**, i.e. $\forall\,(n-1)$-face $F \ni 0^n$, the restriction of c' to F is in $C^{n-1}(X)$.
   Then there is a cube $c \in C^n(X)$ such that
   $$\forall\, v \in \{0,1\}^n \setminus \{1^n\},\ \ c(v) = c'(v).$$



X is a k-**step** nilspace if completion of $(k+1)$-corners is always unique.

X is a **compact nilspace** if X and every cube set $C^n(X)$ are compact Hausdorff.

There is also a naturally defined **Haar probability measure** μ_n on each cube set $C^n(X)$ of a compact nilspace X.

Examples of nilspaces:

- Any abelian group G can be viewed as a 1-step nilspace.

  The cubes $c \in C^n(G)$ are the maps $c(v) = x + v_1 h_1 + \cdots + v_n h_n$ $(x, h_i \in G)$.

  An important k-step nilspace structure on an abelian group G: the nilspace $\mathcal{D}_k(G)$, known as a **degree-k abelian group**, consisting of the Host-Kra cubes for the filtration $G = G_0 = \cdots = G_k \geq G_{k+1} = \{0\}$.

- Any filtered nilmanifold $(G/\Gamma, G_\bullet)$ can be viewed as a **compact** nilspace.

  Indeed, if $G_\bullet$ has **degree** k, and $C^n(G_\bullet)$ are the corresponding groups of Host-Kra cubes for $n \in \mathbb{N}$, then the nilmanifold $X = G/\Gamma$ together with the cubes $C^n(X) = C^n(G_\bullet)/[C^n(G_\bullet) \cap \Gamma^{\{0,1\}^n}]$ is a k-**step compact nilspace**.

  2-step example: the Heisenberg nilspace $\left(\mathcal{H}/\Gamma = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} / \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix},\ \big(\mathrm{C}^n(\mathcal{H}/\Gamma)\big)_{n \geq 0}\right)$.

- **Not every compact nilspace is a nilmanifold**.

  However, every compact nilspace X has a natural **action by a nilpotent group** of homeomorphisms compatible with the cubes: the **translation group** $\Theta(\mathrm{X})$.

  $\longrightarrow$ Dynamical systems on nilspaces, known as **nilspace systems** (these include nilsystems). When we add the nilspace Haar measure, these systems become measure-preserving systems.

Definition. $\varphi : X \to Y$ is a **morphism** of nilspaces if for every cube $c \in C^n(X)$, the composition $\varphi \circ c$ is a cube in $C^n(Y)$.

Examples: - continuous homomorphisms between compact abelian groups.

- polynomial maps $P : \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ of degree at most $k-1$. (These are nilspace morphisms from $\mathcal{D}_1(\mathbb{F}_p^n)$ to $\mathcal{D}_k(\mathbb{R}/\mathbb{Z})$.)

Definition. $\chi : G \to \mathbb{C}$ is a **nilcharacter** of order k (G cpct. ab. group) if

$$\chi = F \circ \varphi, \quad \text{where}$$

- $\varphi : G \to X$ is a morphism of compact nilspaces,
- X is a compact k-step nilspace of *finite rank*,
- $F : X \to \mathbb{C}$ is continuous with absolute value $|F(x)| \leq 1 \ \forall x \in X$.

(Usually, some control on the Lipschitz norm of F and the "complexity" of X is added, and we then say that χ has "bounded complexity".)

Examples: - Fourier characters in $\widehat{G}$ $\;(X = \mathbb{T},\ \varphi \in \mathrm{hom}(G, X),\ F(x) = e^{2\pi i x})$.

- polynomial phase functions $e(P(x))$, $P : \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ polynomial.
- nilsequences $x \in \mathbb{Z} \ \mapsto \ F(h^x\Gamma)$, $\ F : H/\Gamma \to \mathbb{C}$, $\ h \in H$.

More recent inverse theorems for Gowers norms

Qualitative setting: in the nilspace approach, Szegedy had obtained in 2012 inverse theorems in terms of nilspaces for several families of compact abelian groups (beyond the $\mathbb{Z}_N$ and $\mathbb{F}_p^n$ settings). More recently, using the framework of **cubic couplings**, this was extended into the following general result.

Inverse theorem (C., Szegedy 2019). Let $k \in \mathbb{N}$. $\forall \epsilon > 0$, $\exists M > 0$ s.t. if G is a compact abelian group or a nilmanifold, and f is a Borel function on G with $|f| \leq 1$ and $\|f\|_{U^{k+1}} \geq \epsilon$, then there is a nilcharacter $\chi : G \to \mathbb{C}$ of order k and complexity $\leq M$ such that $|\langle f, \chi \rangle| \geq \epsilon^{2^{k+1}}/2$.

In particular, the Green–Tao–Ziegler inverse theorem on $\mathbb{Z}_N$, and the Tao–Ziegler inverse theorem on $\mathbb{F}_p^n$, both follow from this inverse theorem (C.–Szegedy and C.–González-Sánchez–Szegedy).

Quantitative setting: highly non-trivial works of Manners (2018), and of Gowers-Milićević (2021), have produced effective versions of the inverse theorem, respectively for $\mathbb{Z}_N$ and $\mathbb{F}_p^n$ (high-characteristic case), with good bounds.

Some open directions in this area

The Jamneshan–Tao conjecture (2021): in the inverse theorem for **finite** abelian groups, the nilspace X involved in the correlating nilcharacter can be ensured to be a **nilmanifold** H/Γ.

This conjecture has a wide scope, and it also motivates further development of nilspace theory.

Vector spaces $\mathbb{F}_p^n$, p fixed, n increasing.		Groups $\mathbb{Z}/N\mathbb{Z}$, N prime increasing
✓ ———	?	——— ✓
Tao–Ziegler C.–Szegedy–G-Sánchez Gowers-Milićević		Green–Tao–Ziegler C.–Szegedy Manners

Some open directions in this area

Recent progress toward the Jamneshan – Tao conjecture:

- Works of Jamneshan – Shalom – Tao (2023) confirm the conjecture for $\|\cdot\|_{U^{k+1}}$ on abelian groups of **bounded torsion**, with the nilmanifold actually being a degree-d abelian group (polynomial phases of degree at most $d-1$), albeit with d possibly larger than the expected value k.
- C. – González-Sánchez – Szegedy (2023):
  the nilspace X can be ensured to be a **double-coset nilspace** $K\backslash G/\Gamma$, with G a nilpotent Lie group, albeit not ensuring always a nilmanifold. The idea that such a general double-coset representation of nilspaces might hold originated in work of Gutman–Manners–Varjú from 2014 (personal communication).

Quantitative directions: in the recent progress by Manners and Gowers–Milićević for bounds in the inverse theorem, the approaches in the corresponding two settings ($\mathbb{Z}_N$ and $\mathbb{F}_p^n$) are different. The qualitative general inverse theorem unifies the two settings to some extent, at least conceptually. Is there a general quantitatively effective proof of the inverse theorem on all finite abelian groups?

Thank you!