Minicourse on nilspaces, part II: Limit theories and regularization

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Limits of discrete structures

Goal: The goal of limit theories is to study the large scale structure of discrete structures such as graph, hypergraphs, subsets in finite abelian groups, permutations, 0-1 sequences etc...

Method: Let $\mathcal{S}$ be a family of objects as above. Introduce some similarity metric d on the elements of $\mathcal{S}$. Take the completion $\overline{\mathcal{S}}$ of $\mathcal{S}$ with respect to d. Use tools from analysis on $\overline{\mathcal{S}}$ to prove statements about structures in $\mathcal{S}$.

An old example: Furstenberg's correspondence principle can be looked at as a limit language for 0-1 sequences. Limit objects are shift invariant measures on $\{0,1\}^{\mathbb{Z}}$.

Some history: A systematic study of structural limits was started around 2000. Benjamini and Schramm introduced a limit concept for bounded degree graphs. A similar theory for dense graphs was first described in the paper "Limits of dense graph sequences", 2005, Lovász, Sz. and later continued by many authors.

Similarity via Configuration densities

Def: A **graph homomorphism** is a map from the vertex set of a graph H to the vertex set of another graph G such that the induced map on pairs of vertices takes edges to edges.

Def: The homomorphism density $t(H, G)$ is defined to be the probability that a random map from $V(H)$ to $V(G)$ is a graph homomorphism.

Similarity: Homomorphism densities can be used to define a similarity notion for finite graphs. Informally two graphs are similar if their small subgraph densities are similar. This can be metrized as follows:

$$d(G, F) = \sum_{i=1}^{\infty} 2^{-i} |t(H_i, G) - t(H_i, F)|.$$

Dense and sparse graphs

Problem: Graphs with a sub-quadratic number of edges (also called sparse graphs) are similar to the empty graph. Limit theory is essentially blind to sparse graphs.

Solution: For very sparse (bounded degree) graphs one can use a completely different metric (Benjamini-Schramm metric). Two graphs are similar if isomorphism types of small neighborhoods of random points have similar distributions. This can also be expressed by a different notion of subgraph density.

Question: What about graphs in between dense and bounded degree? This intermediate case is an area of active research.

Hypergraphs

Def: A k-**uniform directed hypergraph** is a subset T of V^k where V is the vertex set. If T is symmetric under the natural action of S_k on V^k then the hypergraph is **undirected**.

Homomorphism densities in k uniform hypergraphs can be similarly defined as in graphs. This leads to a similar metrization of k uniform hypergraphs for every fixed k.

Important: Here we don't compare hypergraph of different uniformity and thus there is a separate limit theory for every fixed k

Densities in additive combinatorics and corresponding limits

Def: An additive structure is a pair $S = (A, T)$ where A is a finite Abelian group and T is a subset in A.

Questiuon: What is the analogue of a subgraph density in such a structure?

Answer: The best analogue is the density of a linear configuration such as arithmetic progressions, etc...

A linear configuration C in general is a finite subset in $\mathbb{Z}^k$ for som $k \in \mathbb{N}$.

Def: The density $t(C, S)$ of a linear configuration $C \subset \mathbb{Z}^k$ in an additive structure $S = (A, T)$ is the probability that a random homomorphism $\phi : \mathbb{Z}^k \to A$ takes every element of C into the elements of S.

Remark: Note that $\mathrm{Hom}(\mathbb{Z}^k, A)$ is a finite abelian group and thus it makes sense to talk about a random homomorphism. The notion can be naturally extended to the case where A is compact and thus $\mathrm{Hom}(\mathbb{Z}^k, A)$ is also compact.

Configuration densities in weighted structures

In many limit theories one can extend configuration densities to weighted versions of structures and this will become quite important even in the non-weighted case.

Def: A weighted k-uniform hypergraph is a function of the form $V^k \to \mathbb{R}$ (or $V^k \to \mathbb{C}$). A weighted additive structure is a function $f : A \to \mathbb{C}$ where A is a finite or more generally compact Abelian group. In the general compact case we assume that f is measurable.

Cool and important fact!: homomorphism densities can be extended to the weighted case in the fashion of partition functions in statistical physics. Typically normal (non-weighted) structures have densities in weighted structures. **Somewhat more precisely:** Take a random map from the ground set of the configuration to the ground set of the weighted structure and take the expected value of the product of the weights of images of the elements in the configuration.

Configuration densities: The formulas

Def (hypergraph case): Assume that $f : V^k \to \mathbb{C}$ and $T \subset F^k$.

$$t(T, f) := \mathbb{E}_{\phi : F \to V} \prod_{E \in T} f(\phi^k(E)).$$

Def (additive case): Assume that $f : A \to \mathbb{C}$ and $T \subset \mathbb{Z}^k$.

$$t(T, f) := \mathbb{E}_{\phi : \mathbb{Z}^k \to A} \prod_{E \in T} f(\phi(E)).$$

Norming configurations

Quite surprisingly it turns out that some configurations are special: appropriate power of their densities behave as norms in the weighted case. The simplest such configuration is the four cycle C_4. We have that for a fix V the function

$$\|f\|_{C_4} := t(C_4, f)^{1/4}$$

is a norm on the function space $f : V \times V \to \mathbb{R}$. Note that in the complex case we need to add conjugations to the density formula which is a slight generalization. Another very simple case is the subset

$$C := \{(1,0,1,0),(0,1,1,0),(1,0,0,1),(0,1,0,1)\} \subset \mathbb{Z}^4$$

in the additive case.

$$\|f\|_{U_2} := t(C, f)^{1/4}$$

is the U_2 uniformity norm of Gowers. Again, in case of complex valued f we need to add appropriate conjugations to the formula.

Norming configurations

The C_4 norm for graphs and the U_2 norm for Abelian groups are closely connected:

$$\|f\|_{U_2} = \|f'\|_{C_4}$$

where $f' : A \times A \to \mathbb{R}$ is defined by

$$f'(x, y) := f(x + y).$$

There is a similar situation for hypergraphs and higher Gowers norms. Let $H_k := \{0, 1\}^k$. Then the so-called octahedral norm (Gowers) of a function $f : V^k \to \mathbb{R}$ is defined by

$$\|f\|_{O_k} := t(H_k, f)^{1/2^k}.$$

Furthermore if A is a finite abelian group, $f : A \to \mathbb{R}$ and $f'(x_1, x_2, \ldots, x_k) = f(x_1 + x_2 + \cdots + x_k)$ then

$$\|f\|_{U_k} = \|f'\|_{O_k}$$

where U_k is the k-th Gowers norm. This creates an interesting connection between hypergraphs and additive combinatorics.

Szemerédi's regularity lemma

One of the most celebrated and powerful tools in combinatorics is **Szemerédi's famous regularity lemma**.

A weak version (weaker than the original) of it can be stated using the previous C_4 norm as follows: *For every ϵ there is some constant $F(\epsilon)$ such that if $E \subset V \times V$ is an arbitrary graph on an arbitrary finite vertex set V then its characteristic function 1_E can be ϵ-approximated in $\|.\|_{C_4}$ by a step function $g : V \times V \to [0,1]$ with at most $F(\epsilon)$ steps.*

Remark: There is also a very strong version (stronger than the original) but which involves a very small $\|.\|_{C_4}$-error and an ϵ error in L_2 but it is more technical to state.

Takeaway of this slide: Graphs can be decomposed into a bounded complexity part plus some random noise. This dichotomy between noise and structure can be best expressed and measured by certain norms that come from configuration densities.

Generalizations of Szemerédi's regularity lemma

Question: Can Szemerédi's regularity lemma be generalized to hypergraphs and additive structures using octahedral norms and uniformity norms?

Answer: Yes, but it is very complicated. There are various approaches by many authors. The hypergraph case was settled independently by Gowers and a group of researchers: Rödl, Skokan, Nagle, Schacht around 2000. Few years later Elek and Sz. gave a simpler but non-standard approach which was based on ultra-products of measure spaces. It basically used a limiting point of view and a certain description of factors of ultraproduct spaces. The method is interestingly related to characteristic factors in ergodic theory.

The additive case is even more complicated. The topic is also known as **Higher Order Fourier Analysis** and huge efforts were devoted to solving it in the past 2 decades. Inverse and regularity theorems are given to the Gowers norms in great generality however the topic still has many open problems. The non-standard approach also proved to be quite useful in this case. Efforts led to the systematic algebraic study of certain structures called **nilspaces**

The Non-standard approach

Basics: Let ω be a non-principal ultrafilter and let $\{S_i\}_{i\in\mathbb{N}}$ be finite sets. The set $S = \prod_\omega S_i$ is a measure space in the following way. The σ-algebra is generated by sets of the form $S' := \prod_\omega S_i'$ where $S_i' \subseteq S_i$. We set $\mu(S') := \lim_\omega |S_i'|/|S_i|$.

An illustrative application: Let $G_i = (V_i, E_i)$ be a sequence of finite graphs with $E_i \subseteq V_i \times V_i$. Let $V := \prod_\omega V_i, E := \prod_\omega E_i$. We have that $V \times V = \prod_\omega (V_i \times V_i)$ and $E \subseteq V \times V$.

important fact: The σ-algebra on $V \times V$ is not the same as the product of the two σ-algebras on its two components. There are 4 interesting σ-algebras: $\mathcal{A}_{1,2}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_1 \vee \mathcal{A}_2$.

Szemerédi's regularity lemma: The formula

$$\mathbb{E}(1_E | \mathcal{A}_1 \vee \mathcal{A}_2)$$

is a measurable function on $V \times V$ that takes values in $[0, 1]$. It has a step function approximations with finite number of steps.

The Non-standard approach

Seminorm characterization: A bounded measurable function $f : V \times V \to \mathbb{R}$ satisfies

$$\|f\|_{C_4} = \|\mathbb{E}(f|\mathcal{A}_1 \vee \mathcal{A}_2)\|_{C_4}.$$

In particular C_4 norm becomes a semi-norm on $V \times V$.

Seminorm <—> sub σ-algebra

The σ-algebra $\mathcal{A}_1 \vee \mathcal{A}_2$ is generated by the functions that are orthogonal to every function g with $\|g\|_{C_4} = 0$.

Decomposition into structured and random parts: We have a unique decomposition $f = f_s + f_r$ where $f_s = \mathbb{E}(f|\mathcal{A}_1 \vee \mathcal{A}_2)$ and $f_r = f - f_s$. This decomposition is unique with the property that $\|f_r\|_{C_4} = 0$ and f_s is orthogonal to every function g with $\|g\|_{C_4} = 0$.

The Non-standard approach

Approach to hypergraph regularity: The set V^k has many interesting σ-algebras generated by $2^k - 1$ "marginal" σ-algebras. This gives the algebraic difficulty behind the hypergraph regularity lemma. The non-standard approach leads to a significantly simplified and more conceptual proof.

Approach to Higher Order Fourier Analysis: Let A be the ultra product of finite Abelian groups $\{A_i\}_{i=1}^{\infty}$.

1. Each Gowers norm becomes a semi-norm on A.
2. To each Gowers norm $\|.\|_{U_k}$ there is a corresponding shift-invariant σ-algebra $\mathcal{F}_{k-1}$.
3. $L^2(\mathcal{F}_k)$ is the orthogonal sum of rank one, shift invariant $L^\infty(\mathcal{F}_{k-1})$ modules. These modules form the elements of the k-th order dual group of A.

The Non-standard approach

Non-standard regularity lemma for the U_k-norm: If $f \in L^\infty(A)$ then $f = f_s + f_r$ such that f_s is measurable in $\mathcal{F}_{k-1}$ and $\|f_r\|_{U_k} = 0$. Furthermore there exists a "continuous" morphism $\tau : A \to N$ to a $k-1$-step compact nilspace, and a measurable map $\phi : N \to \mathbb{R}$ such that $f_s = \phi \circ \tau$.

Note: Nilspaces are special algebraic structures that are presheaves over the category of Abelian groups with some extra condition.

Limit theories

Back to original question: What are limits of graphs, hypergraphs and additive structures? Dense graph, hypergraph case is done (Lovász-Sz, Elek, Sz: graphons, hypergraphons)

Additive limits are completely described for the groups Z_2^n by nilspace methods (exchangeability result Candela, González-Sánchez, Sz) and partially described for general Abelian groups (measurable functions on nilspaces)

Cool algebraic observation: Preshevaes appear in these limit theories. Hypergraphs —> Delta complexes, Additive structures —> Nilspaces

Remarks on ergodic theory

Host-Kra seminorms and Gowers norms are very closely related. The Gowers semi-norms on ultra product groups play a very similar role as the Host-Kra seminorms. Furthermore the Fourier σ-algebras $\mathcal{F}_k$ play a similar role as characteristic factors in ergodic theory.

Further connection is found through some measure theoretic constructions called cubic couplings (Candela-Sz.) Nilspace theory also found applications in ergodic theory.