Minicourse on Nilspaces (Part III)
Nilspaces in Topological Dynamics

Nilpotent Structures in Topological Dynamics, Ergodic Theory and Combinatorics

Będlewo, Poland, June 5th, 2023

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Structure theorems for t.d.s

- Let (G, X) be a topological dynamical system (t.d.s). That is:
- X is a compact (Hausdorff) space. G is topological (Hausdorff) group. Action denoted by $g.x$ for $g \in G$ and $x \in X$.
- G acts on X, $G \curvearrowright X$: $e.x = x$, $g.(h.x) = (gh).x$
- *Oftentimes* (as well as in the sequel) G and X are assumed metric.
- Motivating question: What is the structure of t.d.s?
- Standing assumption: (G, X) is minimal, that is every **orbit**, $G.x \triangleq \{g.x | g \in G\}$ is dense.
- Known structure theorems: Fustenberg (1963), Ellis-Glasner-Shapiro (1975),Veech (1977),...

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Equicontinuous systems

Definition

(G, X) is equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$ so that for every $g \in G$, $x_1, x_2 \in X$

$$d(x_1, x_2) < \delta \Rightarrow d(g.x_1, g.x_2) < \epsilon$$

Examples

- Irrational rotation on the circle.
- $SO_n(\mathbb{R})$ acts on $\mathbb{R}^n \supseteq \mathbb{S}^{n-1} = {SO_n(\mathbb{R})}/{SO_{n-1}(\mathbb{R})}$.

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Structure theorem for minimal equicontinuous t.d.s

Theorem

(G, X) is minimal equicontinuous iff $X = K/H$ is a homogeneous space, where K is a compact group, H is a closed subgroup and G acts through a continuous group homomorphism with dense image $\phi : G \to K$, $g.kH = \phi(g)kH$.
In particular if G is abelian, K is abelian (and w.l.o.g $H = \{0\}$).

Question: Can a general minimal t.d.s be "reduced" to a minimal equicontinuous t.d.s?

Definition

(G, Y) is a factor of (G, X) if there exists a surjective continuous map $\phi : (G, X) \to (G, Y)$ which is G-equivariant, e.g.,

$$\forall g \in G, x \in X \ \phi(g.x) = g.\phi(x)$$

It is not hard to show that there is a maximal equicontinuous factor, however how to concretely characterize it?

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Relations in dynamics

Definition

A **relation** over X, $\mathbf{R} \subset X \times X$ is called:

- closed if $\mathbf{R}$ is closed in $X \times X$.
- G-invariant if $(x, y) \in \mathbf{R}$ implies $(g.x, g.y) \in \mathbf{R}$ for all $g \in G$.
- reflexive if $(x, x) \in \mathbf{R}$ for all $x \in X$.
- symmetric if $(x, y) \in \mathbf{R}$ implies $(y, x) \in \mathbf{R}$.
- transitive if $(x, y) \in \mathbf{R}$ and $(y, z) \in \mathbf{R}$ imply $(x, z) \in \mathbf{R}$.
- an equivalence relation if it is reflexive, symmetric and transitive.

Let $(G, X) \to (G, Y)$ be a factor map. Define a closed G-invariant equivalence relation $\mathbf{R} \subset X \times X$ by $(x, y) \in \mathbf{R}$ iff $\phi(x) = \phi(y)$. Then $Y = X/\mathbf{R}$.

Conversely if $\mathbf{R} \subset X \times X$ is a closed G-invariant equivalence relation then $(G, X) \to (G, X/\mathbf{R})$ is a factor map.

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The regionally proximal relation

Definition (Ellis & Gottschalk 1960)

$x, y \in X$ are regionally proximal, denoted $(x,y) \in \mathbf{RP}(X)$, if $\exists g_i \in G,\ x_i, y_i \in X$

$$x_i \to x,\ y_i \to y,\ (g_i x_i, g_i y_i) \to \triangle_X \triangleq \{(x,x) | x \in X\}$$

Theorem (Ellis & Gottschalk 1960)

The smallest closed G-invariant equivalence relation which contains $\mathbf{RP}(X)$ corresponds to the maximal equicontinuous factor of (G, X).

$\mathbf{RP}(X)$ is closed G-invariant, reflexive and symmetric but not necessarily transitive. Thus sometimes $\mathbf{RP}(X)$ is not an equivalence relation.

Theorem (Veech 1968)

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Cube groups

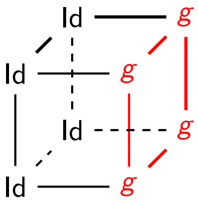
Let G be a group. Define:

$$G^{[d]} \triangleq G^{\{0,1\}^d}$$

Definition

Let F be a hyperface of the discrete cube $\{0,1\}^d$. For $g \in G$ define $g^F \in G^{[d]}$:

$$g^F(\epsilon) = \begin{cases} g & \epsilon \in F \\ \mathrm{Id} & \epsilon \notin F \end{cases}$$



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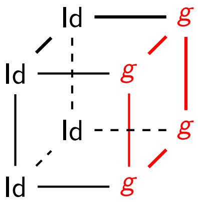

Host-Kra cube groups

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$$\mathcal{HK}^{[d]}(G) = \langle g^F \,|\, F \text{ is a hyperface of } \{0,1\}^d,\ g \in G \rangle \subset G^{[d]}$$

Example

$G = \mathbb{Z}$, $d = 2$.

$$\mathcal{HK}^{[2]}(\mathbb{Z}) = \left\{ \begin{matrix} c & d \\ a & b \end{matrix} \,\middle|\, a, b, c, d \in \mathbb{Z},\ a - b - c + d = 0 \right\}$$

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Dynamical cubespace

Let (G, X) be a t.d.s. Define the (G, X)-dynamical cubespace $(X, C_G^\bullet)$ by:

$$C_G^{[d]}(X) = \overline{\{\mathcal{HK}^{[d]}(G).(x, x, \dots, x) | x \in X\}} \subset X^{[d]}, \ d \in \mathbb{N}$$

Note $(\mathcal{HK}^{[d]}(G), C_G^{[d]}(X))$ is a t.d.s (action is coordinate-wise).

Example

$X = \mathbb{S}^1$, $G = \mathbb{Z}$, $x \mapsto x + \alpha$ (α irrational), $d = 2$.

$$C_{\mathbb{Z}}^{[2]}(X) = \left\{ \begin{matrix} c\ulcorner & \urcorner d \\ a\llcorner & \lrcorner b \end{matrix} \,|\, a, b, c, d \in \mathbb{S}^1, \ a - b - c + d = 0 \right\}$$

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$$(x, y) \in \textbf{NilRP}^{[d]}(X) \overset{\triangle}{\Leftrightarrow} (x, x, \ldots x, y) \in C_G^{[d+1]}(X)$$

Definition of $\mathbf{NilRP}^{[d]}(X)$ ([1,4])

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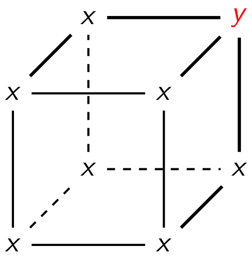

Fibrant dynamical cubespaces [1,4]

Proposition

There exists a $\mathbb{Z}$ minimal system whose dynamical cubespace is not fibrant.

Definition

Let (G, X) a topological dynamical system. We call (G, X) distal, if for any $x, y \in X$ and any sequence $\{g_n\}_{n=1}^{\infty} \subset G$, $\lim_{n\to\infty} g_n x = \lim_{n\to\infty} g_n y$ implies $x = y$.

Examples

Examples of distal systems include equicontinuous systems and nilsystems. The class is closed under inverse limits and isometric extensions.

Theorem

Let (G, X) be a minimal distal topological dynamical system, then the cubespace $(X, C^{\bullet}_{G})$ is ergodic and fibrant.

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There exists a $\mathbb{Z}$ minimal system whose dynamical cubespace is not fibrant.

Definition

Let (G, X) a topological dynamical system. We call (G, X) distal, if for any $x, y \in X$ and any sequence $\{g_n\}_{n=1}^{\infty} \subset G$, $\lim_{n\to\infty} g_n x = \lim_{n\to\infty} g_n y$ implies $x = y$.

Examples

Examples of distal systems include equicontinuous systems and nilsystems. The class is closed under inverse limits and isometric extensions.

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$\mathbf{NilRP}^{[k]}(X)$: basic properties

The relation $\mathbf{NilRP}^{[k]}(X)$ is:

- $\mathbf{RP}(X) \subseteq \mathbf{NilRP}^{[1]}(X) \subseteq \cdots \subseteq \mathbf{NilRP}^{[k]}(X) \subseteq \cdots$.
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Is $\mathbf{NilRP}^{[k]}(X)$ *an equivalence relation?*

Structure theorems for $\textbf{NilRP}^{[d]}(X)$ [1-5]

Theorem

Let (G, X) be a minimal t.d.s then $\textbf{NilRP}^{[d]}(X)$ *is an equivalence relation.*

Theorem

Let (G, X) be a minimal t.d.s and suppose G has a dense compactly generated subgroup, then $(G, X/\textbf{NilRP}^{[d]}(X))$ is the d-th maximal pronilfactor of (G, X), i.e.:

- $X_d \triangleq X/\textbf{NilRP}^{[d]}(X)$ *is isomorphic to an inverse limit of nilsystems*

$$\varprojlim (G, L_n/\Gamma_n)$$

- *L_n is a d-step nilpotent Lie group, Γ_n is cocompact discrete and G acts through a continuous group homomorphism $\phi : G \to L_n$, $g.\ell\Gamma_n = \phi(g)\ell\Gamma_n$.*
- *X_d is the maximal factor w.r.t to these properties.*

Open Question [4]

Question

For which groups G, is $(G, X/\,\boldsymbol{NilRP}^{[d]}(X))$ the d-th maximal pronilfactor of (G, X)

Note that for $d = 1$, $(G, X/\,\mathbf{NilRP}^{[1]}(X))$ is the maximal compact abelian group factor of (G, X).

Theorem

Let $d \geq 1$ and let (G, X) be a minimal topological dynamical system, then $(G, X/\,\boldsymbol{NilRP}^{[d]}(X))$ is the maximal factor which is a nilspace of order at most d.

Gluable cubes:

Gluable Cubes [3]

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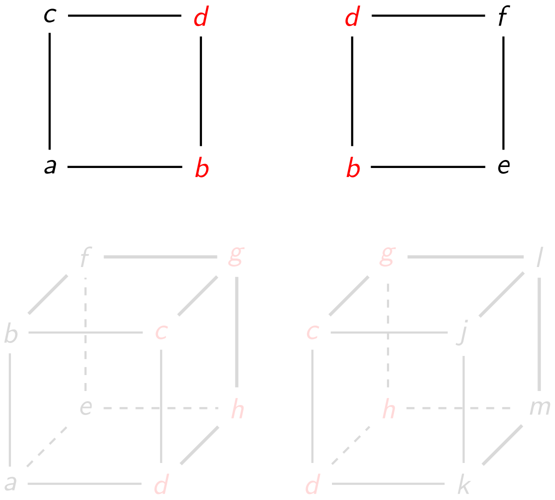

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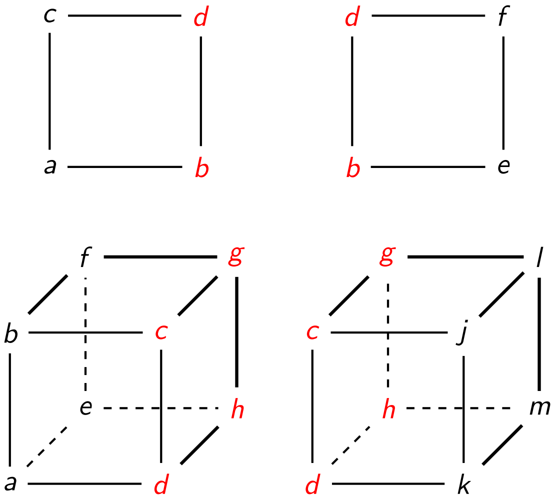

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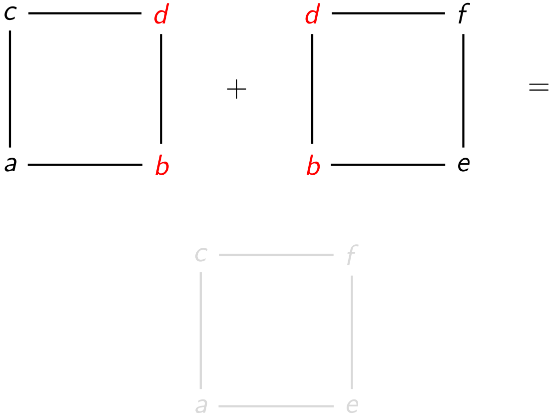

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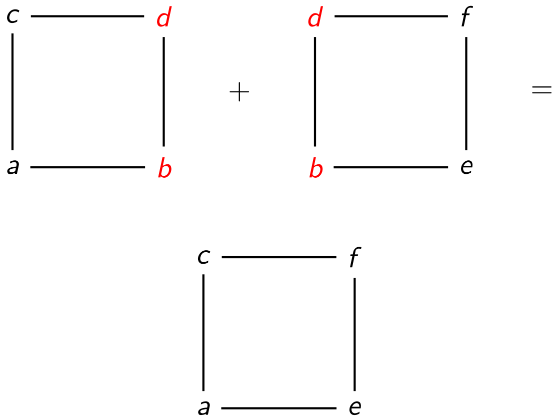

Gluing implies Symmetry

Definition

(G, X) has gluing if for all $n \in \mathbb{N}$ $c_1, c_2 \in C_G^{[n]}(X)$ whenever c_1, c_2 are gluable, $c_1 + c_2 \in C_G^{[n]}(X)$.

Reduction to gluing:

Lemma

If (G, X) has gluing, then $\mathbf{NilRP}^{[k]}(X)$ is symmetric.

$(x, y) \in \mathbf{NilRP}^{[k]}(X) \overset{?}{\Rightarrow} (y, x) \in \mathbf{NilRP}^{[k]}(X)$

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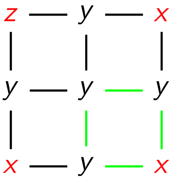
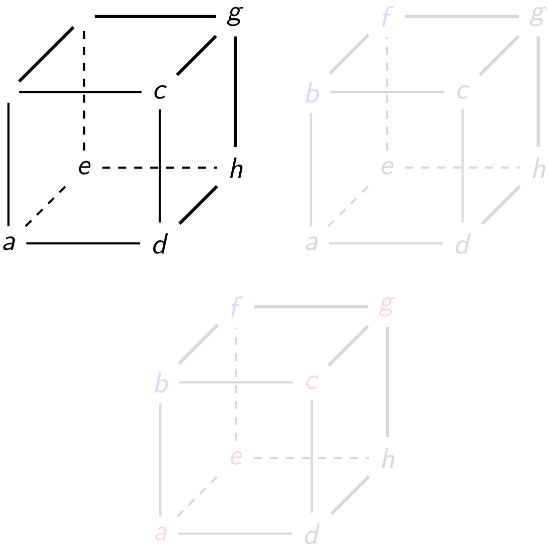


Extension implies Gluing

Gluing achieved by extension and projection:



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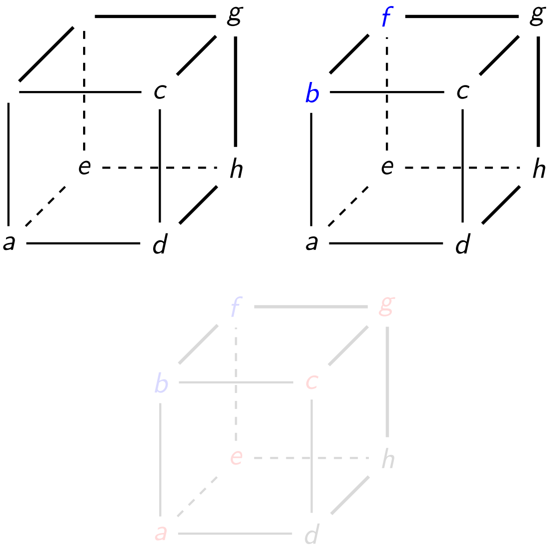

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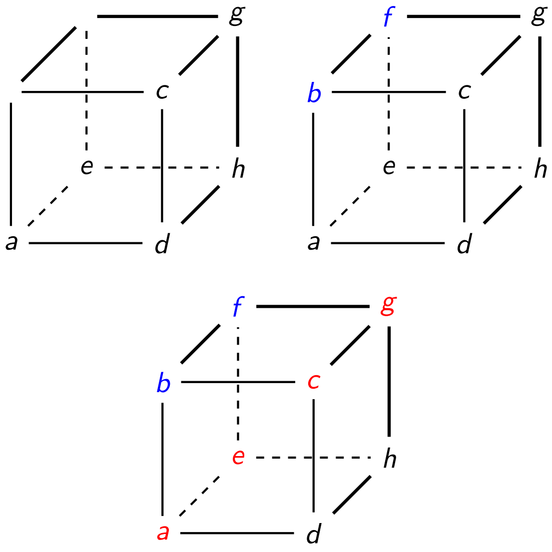

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A fibrant cubespace has the extension property.

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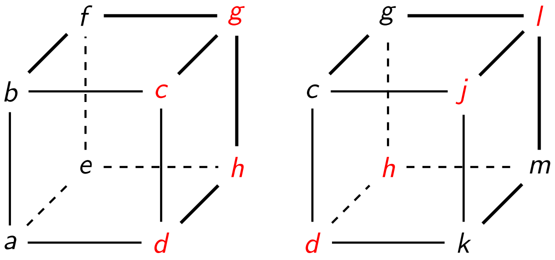
Projections & Partial Ordering

Given $c \in C^3_G(X)$ various projections give rise to 2-cubes, e.g.:



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Partial order on $\{0,1\}^n$ $\leq$: $\vec{v} \leq \vec{w}$ if $v_i \leq w_i$ for all i. E.g. $0010 \leq 0110$. $S \subset \{0,1\}^n$ is a downwards-closed subset if closed under $\leq$. $\epsilon \in \{0,1\}^n$ is maximal if maximal w.r.t $\leq$.

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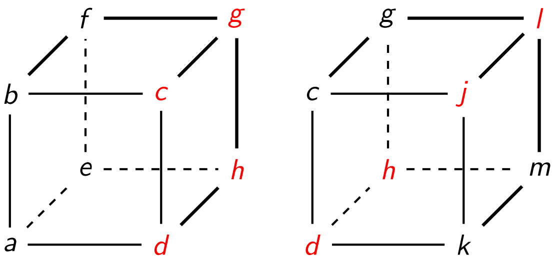

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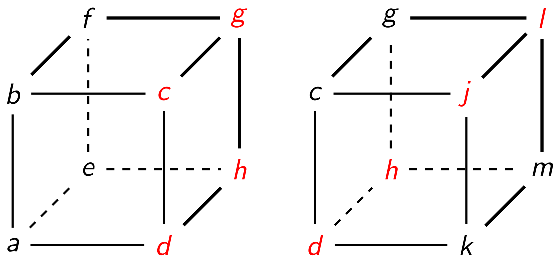

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Morphisms and Extension

$Hom_G(S, X)$

Let $S \subset \{0,1\}^n$ be a downwards-closed subset. $f : S \to X$ is a morphism if for every $D \subset S$ k-dimensional face of $\{0,1\}^n$ contained in S, $f_{|D} \in C^k_G(X)$.

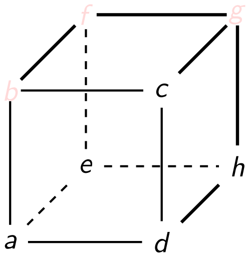
Definition

(G, X) has extension if any morphism $f : S \to X$, $S \subset \{0,1\}^n$ extends to $c \in C^n_G(X)$, i.e. $c_{|S} = f$.

Morphisms and Extension

$Hom_G(S, X)$

Let $S \subset \{0,1\}^n$ be a downwards-closed subset. $f : S \to X$ is a morphism if for every $D \subset S$ k-dimensional face of $\{0,1\}^n$ contained in S, $f_{|D} \in C^k_G(X)$.



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(G, X) has extension if any morphism $f : S \to X$, $S \subset \{0,1\}^n$ extends to $c \in C^n_G(X)$, i.e. $c_{|S} = f$.

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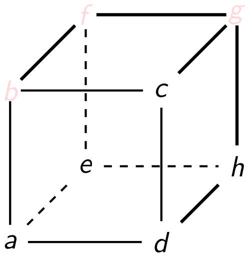

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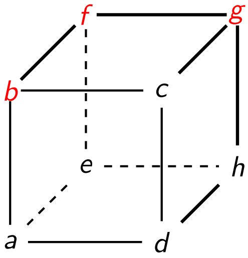

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Establishing Extension I [4]

- Proof by induction. In particular assume true for downwards-closed subsets in $\{0,1\}^{n-1}$
- Let $f : \mathcal{V} \to X$, $\mathcal{V} \subset \{0,1\}^n$ be a morphism.
- If $|\mathcal{V}| = 1$, then f is extendable.
- Assume $|\mathcal{V}| \geq 2$ and $\mathcal{V} \neq \{0,1\}^n$. Let $\epsilon \in \mathcal{V}$ be a maximal element in $\mathcal{V}$ and denote $\mathcal{W} = \mathcal{V} \setminus \{\epsilon\}$ (note $|\mathcal{W}| \geq 1$).
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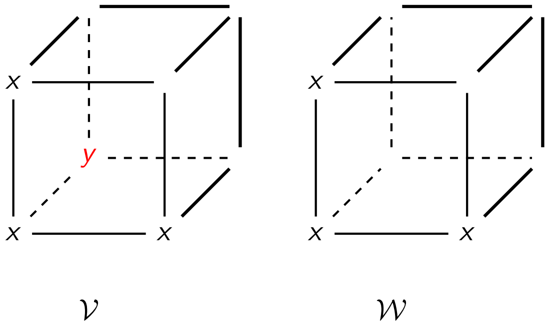

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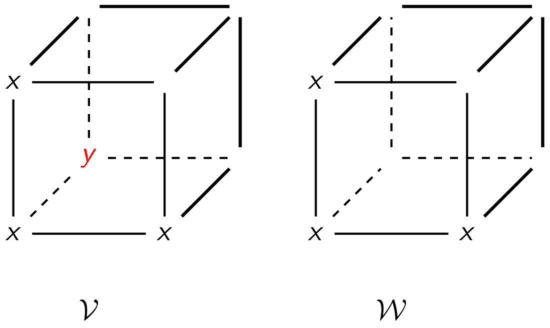

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- Let $\mathcal{F}$ be a lower face of $\{0,1\}^n$ which contains ϵ.
- By the inductive assumption $f_{|\mathcal{V}\cap\mathcal{F}}$ is extendable to $\mathcal{F}$. Let us call the extension h.
- Let $\mathcal{H}$ be the upper face which is parallel to $\mathcal{F}$.
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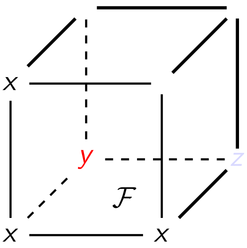

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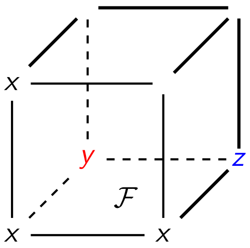

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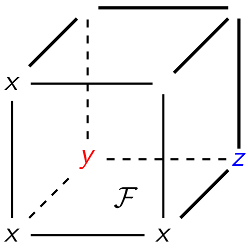

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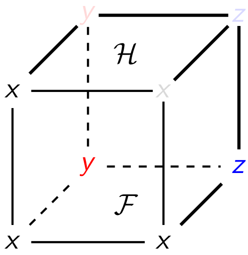

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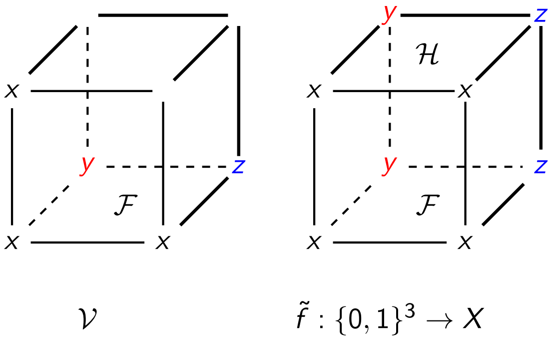

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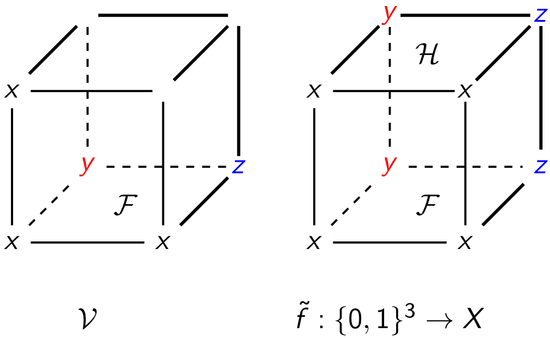

Using Distality I

The general case:

Lemma

If (G, X) is a distal minimal t.d.s then $(\mathcal{HK}^{[n]}(G), C_G^{[n]}(X))$ is a distal minimal t.d.s.

Proof: As $(G^{[n]}, X^{[n]})$ is distal so is $(\mathcal{HK}^{[n]}(G), C_G^{[n]}(X))$. As $C_G^{[n]}(X) = \overline{\{\mathcal{HK}^{[n]}(G).(x, x, \ldots, x) | x \in X\}} = \overline{\mathcal{HK}^{[n]}(G).(x_0, x_0, \ldots, x_0)}$ is dynamically transitive it is minimal.

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Distal exchangablility

Let (G, X) be distal. Then $x \in \overline{Gy}$ iff $y \in \overline{Gx}$.

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The Ellis enveloping semigroup

It would be very convenient if we could "compactify" (G, X)...
For example, for any $g_1, g_2, \ldots \in G$, one would be able to find a subsequence g_{i_j} such that $g_{i_j}x$ converges for all $x \in X$.

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The **Ellis (enveloping) semigroup** $E = E(G, X)$ of a t.d.s (G, X) is the closure of G in the semigroup (with respect to composition) X^X equipped with the product topology.

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