Structure theorems for the Host–Kra characeristic factors and inverse theorems for the Gowers uniformity norms

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The Gowers uniformity norms

Let $G = (G, +)$ be a **finite** abelian group. We can represent

$$G = \mathbb{Z}/p_1^{\beta_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_k^{\beta_k}\mathbb{Z}$$

for not necessarily distinct primes $p_1, \ldots, p_k$. Let $f : G \to \mathbb{C}$. The Gowers uniformity norms are defined by the formula

The Gowers norms

$$\|f\|_{U^k(G)} = \left(\mathop{\mathbb{E}}_{h_1,\ldots,h_k,x \in G} \partial_{h_1} \ldots \partial_{h_k} f(x) \right)^{1/2^k}$$

where $\partial_h f(x) = f(x+h) \cdot \overline{f(x)}$.

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where $\partial_h f(x) = f(x+h) \cdot \overline{f(x)}$.

For instance, when $k = 2$ we get

$$\|f\|_{U^2(G)} = \left(\mathop{\mathbb{E}}_{h_1,h_2,x\in G} f(x) \cdot \overline{f(x+h_1)} \cdot \overline{f(x+h_2)} \cdot f(x+h_1+h_2) \right)^{1/4}.$$

The inverse problem

We are interested in the inverse problem: If $f : G \to \mathbb{C}$ is 1-bounded ($|f(x)| \leq 1$ for all $x \in G$) and

$$\|f\|_{U^k(G)} \geq \delta > 0$$

is large, then what can we say about f?

The inverse conjecture for the Gowers uniformity norms predicts that a large Gowers norm indicates significant correlation with a function of an algebraic origin.

Let $\Gamma = (\Gamma, +)$ be a **countable** abelian group.

Γ-system

A Γ-system is a probability space $\mathrm{X} = (X, \mathscr{X}, \mu)$, equipped with a measure-preserving action $T : \Gamma \to \mathrm{Aut}(\mathrm{X})$.

The Gowers-Host-Kra seminorms

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For ergodic Γ-system X and $f \in L^\infty(\mathrm{X})$, we define

Host-Kra seminorms

$$\|f\|_{U^k(\mathrm{X})} = \lim_{N_1,\ldots,N_k\to\infty} \left(\mathop{\mathbb{E}}_{h_1,\ldots,h_k\in\Phi_{N_1},\ldots,\Phi_{N_k}} \int_X \partial_{h_k} \ldots \partial_{h_1} f(x)dx \right)^{1/2^k}$$

where $\partial_h f(x) = f(T^h x)\cdot\overline{f(x)}$, and Φ_N is a Følner sequence.

Structure theorem for $\mathbb{Z}$-systems

Universal Characteristic Factors

For any Γ-system, there exist factors

$$X \to \ldots \to Z_k(\mathrm{X}) \to Z_{k-1}(\mathrm{X}) \to \ldots \to Z_0(\mathrm{X})$$

such that $\forall f \in L^\infty(\mathrm{X})$

$$\|f\|_{U^{k+1}(\mathrm{X})} = 0 \iff \mathbb{E}(f|Z_k(\mathrm{X})) = 0.$$

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Structure theorem for $\mathbb{Z}$-systems, Host and Kra 2005

$Z_k(\mathrm{X})$ is an inverse limit of k-step nilsystems.

Example

Let $X = H_3(\mathbb{R})/H_3(\mathbb{Z})$ where

$$H_3(R) = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in R \right\}.$$

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$\mathrm{X} = (X, \text{Haar}, T)$ is an ergodic 2-step nilsystem and $\mathrm{X} = Z_2(\mathrm{X})$.

Green-Tao-Ziegler inverse theorem

- Green and Tao published a paper called "linear equations in primes", where under two conjectures they managed to prove the existence (and asymptotics) of prime solutions to certain linear equations.
- One of these conjecture was a 1-bounded function $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ with large k-Gowers norm correlates with a nilsequence.
- This conjecture was proved later by Green and Tao for $k = 3$ and for general k together with Ziegler.
- Subsequent alternative proofs (including quantitative improvements) where established by Szegedy, Candela–Szegedy, and Manners.

Definition

Let X be a Γ system. A function $f : \mathrm{X} \to S^1$ is called a polynomial of degree k, if $\partial_{\gamma_1} \ldots \partial_{\gamma_{k+1}} f = 1$, for all $\gamma_1, ..., \gamma_{k+1} \in \Gamma$.

Vector spaces over finite fields

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Bergelson-Tao-Ziegler Theorem, 2010

Let X be an ergodic $\mathbb{F}_p^\omega$-system.

- (High characteristics) If $p > k$, then $L^2(Z_k(\mathrm{X}))$ is generated by linear combinations of polynomials of degree k.
- (Low characteristics) If $p \leq k$, then it is generated by linear combinations of polynomials of degree $C(k)$.

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Finite dimensional vector spaces

Inverse theorem for the Gowers norms over finite fields. Tao-Ziegler 2010.

Let p be a fixed prime and let V be a finite dimensional vector space over $\mathbb{F}_p$. For every $\delta > 0$, there exists $\varepsilon > 0$ so that for any 1-bounded map $f : V \to \mathbb{C}$ be 1-bounded with $\|f\|_{U^{k+1}(V)} > \delta$, there exists a polynomial $p : V \to S^1$ of degree k with

$$\left|\mathbb{E}_{x\in V} f(x)\overline{p(x)}\right| > \varepsilon.$$

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Subsequent alternative proofs (including quantitative improvements) by Gowers–Milićević, Milićević, Candela–González-Sánchez–Szegedy.

The Bergelson-Tao-Ziegler conjecture

Bergelson, Tao and Ziegler conjectured that the constant $C(k)$ equals k even when $p < k$. Recently we disproved this conjecture.

Jamneshan, S., Tao, (preprint)

There exists an ergodic $\mathbb{F}_2^\omega$-system X such that $L^2(Z_5(\mathrm{X}))$ is not generated by linear combinations of polynomials of degree 5.

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- It is not obvious that 5 is the least order for which the theorem fails. In 2020 Candela, Gonzáles-Sánchez, Szegedy proved that the conjecture holds whenever $k \leq p+1$.
- What is the optimal value of $C(k)$?

What about other abelian groups?

Jamneshan and Tao's conjecture

Let G be a finite additive group, let $\eta > 0$, let $k \geq 1$, and let $f\colon G \to \mathbb{C}$ be a 1-bounded function with $\|f\|_{U^{k+1}(G)} \geq \eta$. Then there exists a degree k filtered nilmanifold H/Γ, drawn from some finite collection $\mathscr{N}_{k,\eta}$ of such nilmanifolds that depends only on k, η but not on G (and each such nilmanifold in $\mathscr{N}_{k,\eta}$ is endowed arbitrarily with a smooth Riemannian metric), a Lipschitz function $F\colon H/\Gamma \to \mathbb{C}$ of Lipschitz norm $O_{\eta,k}(1)$, and a polynomial map $g\colon G \to H/\Gamma$ such that

$$|\mathop{\mathbb{E}}_{x\in G} f(x)\overline{F(g(x))}| \gg_{\eta,k} 1.$$

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The inverse $U^3(G)$ was recently established by Jamneshan and Tao.

Nonstandard analysis formulation

The proof of the inverse conjectures for Gowers norms using ergodic theory is by contradiction.

The contrapositive gives the existence of some η, k such that there are a sequence of finite abelian groups G_n (in the general class of such groups, or in a subclass such as the finite vector spaces over a fixed field of prime order, all cyclic groups, or groups of uniformly bounded torsion, etc.) and a corresponding sequence of bounded functions $f_n \colon G_n \to \mathbb{C}$ with $\|f_n\|_{U^k(G_n)} \geq \eta$ which **asymptotically** violate the correlation conclusion for certain choices of nilmanifolds (by enumerating nilmanifolds using the theory of Mal'cev bases).

Nonstandard analysis allows one to turn this asymptotic non-correlation into a contradiction to an inverse conjecture for the Gowers norms for hyperfinite abelian groups (imagine them as large asymptotic products of finite abelian groups).

The correspondence principle

On a hyperfinite abelian group, the group $\mathbb{Z}^\omega$ enables an action by coordinate-wise independent translations (a random sampling over an independent coordinate-wise uniform distribution guarantees a "generic" such action almost surely).

This gives an ergodic $\mathbb{Z}^\omega$-action on a probability space such that the Gowers norms for a collection of functions on the hyperfinite abelian group coincide with the Gowers–Host–Kra seminorms of these functions in the $\mathbb{Z}^\omega$-system. Now the contradiction to an inverse conjecture for the Gowers norms for hyperfinite abelian groups contradicts an inverse conjecture for the corresponding Gowers–Host–Kra seminorms in that $\mathbb{Z}^\omega$-system.

Structure theorem for $\mathbb{Z}^\omega$-actions

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$Z_2(X)$ is an inverse limit of 2-step nilpotent translational system.

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- $S_g n\Lambda = \phi(g)n\Lambda$, $\phi : \mathbb{Z}^{\omega} \to N$ a homomorphism.

Rudolph example

There exists a $\mathbb{Z}$-system X with $Z_2(\mathrm{X}) = \varprojlim H_3(\mathbb{R})/\Lambda_n$, where

$$\Lambda_n = \begin{bmatrix} 1 & 2^n\mathbb{Z} & 2^n\mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix}$$

that is NOT isomorphic to a 2-step translational system.

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Yet,

$$Z_2(X) \cong K \backslash H_3(\mathbb{R} \times \mathbb{Z}_2)/H_3(\Delta_{\mathbb{Z}})$$

where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbb{Z}_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Exact structure theorem (S. 2021)

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Conjecture: $Z_k(X)$ is isomorphic to a k-step double-coset system.

Extensions

Let Γ be torsion-free.

- Every ergodic Γ-system X admits an extension whose group of normalized eigenfunctions $\{f : \mathrm{X} \to S^1 : \partial_\gamma f = \lambda_\gamma,\ \text{for } \lambda_\gamma \in S^1\}$ is divisible.
- Let Y be a Γ-system with a divisible group of eigenfunctions, then $Z_2(\mathrm{Y})$ is a translational system.
- A factor of a translational system is a double-coset.

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Let $\Gamma := \mathbb{Z}^\omega$, and let X be a *totally disconnected* ergodic Γ-system.

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Existence of divisible extensions

The system X admits an extension Y such that for every k, the group of polynomials of degree k of Y is divisible.

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Weak BTZ-Conjecture.

Every ergodic $\mathbb{F}_p^\omega$-system X, admits a $\mathbb{Z}^\omega$-extension Y with the property that $L^2(Z_k(\mathrm{Y}))$ is generated by polynomials of degree k.

Inverse theorem for groups of bounded torsion

(Jamneshan, S., Tao. preprint)

Let $k, m \in \mathbb{N}$ be a fixed natural number. There exists a constant $C(k, m)$ such that for every $\delta > 0$, there exists $\varepsilon > 0$ so that for every m-torsion group finite G and every 1-bounded $f : G \to \mathbb{C}$ with $\|f\|_{U^{k+1}(G)} > \delta$ we can find a polynomial $p : G \to \mathbb{C}$ of degree at most $C(k, m)$ so that $\left|\mathbb{E}_{x \in G} f(x) \cdot \overline{p(x)}\right| \geq \varepsilon$.

What about connected systems?

A $\mathbb{Q}$-system and more generally $\mathbb{Q}^\omega$-system is automatically totally ergodic and connected.

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Theorem

Let $k \geq 1$, let $\Gamma = \mathbb{Q}^{\omega}$ and let X be a Γ-system with divisible group of polynomials of degree k. Then $Z_k(\mathrm{X})$ is a k-step translational system. Furthermore, as a nilspace $Z_k(\mathrm{X})$ is an inverse limit of nilmanifolds.

Inverse theorem for approximately torsion-free

Definition

A family $\mathscr{F}$ of finite abelian groups is called *approximately torsion-free* if for every $p \in \mathbb{P}$ there are at most finitely many $G \in \mathscr{F}$ whose order divides p.

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Inverse theorem for approximately torsion-free groups

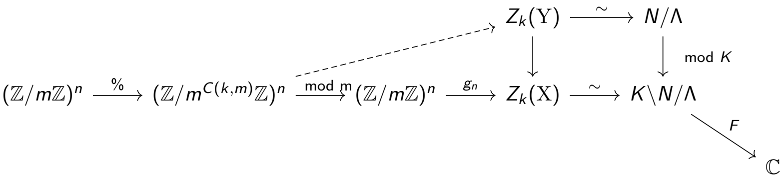
Jamneshan Tao conjecture holds for approximately torsion-free groups.

Outline of the difficulties

A main difficulty in the proof arises from that we are simultaneously working in three different categories: a dynamical, a topological, and a combinatorial.

By the correspondence principle, we get correlation with a "double-nilsequence", that we want to lift to a correlation with a nilsequence (which then turns out to be a polynomial sequence).

A key step in the proof



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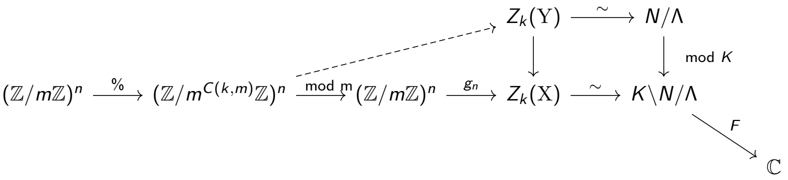

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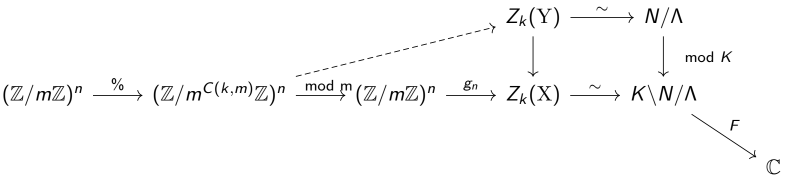

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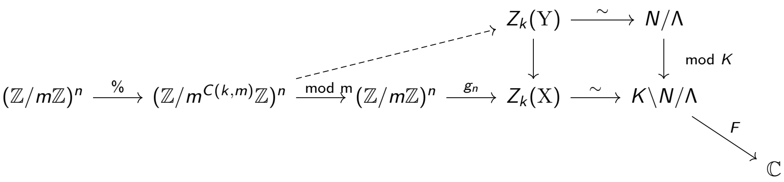

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- Ergodic Sylow theorem + nilspace Schur-Zassenhaus theorem
- Continuous   mod K not a nilspace fibration *a priori*

Thank you for listening!