Joint ergodicity beyond polynomials
(joint with A. Koutsogiannis and W. Sun)

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Nilpotent structures in topological dynamics, ergodic theory and combinatorics

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Definitions

In this talk the setting is:

- $(X, \mathcal{X}, \mu)$ is a probability space.

- $T: X \to X$ is a bi-measurable, measure preserving transformation. This means that $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{X}$.

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T is ergodic for μ (or $(X, \mathcal{X}, \mu, T)$ is ergodic) if for any $f \in L^2(\mu)$,

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \to \int f \, d\mu$$

where the convergence is in $L^2(\mu)$.

Joint ergodicity

Let $(X, \mathcal{X}, \mu, T_1, \ldots, T_d)$ be a measure preserving system and $a_1, \ldots, a_d$ be sequences of integers.
We say that $(T_1^{a_1(n)}, T_2^{a_2(n)}, \ldots, T_d^{a_d(n)})_{n\in\mathbb{N}}$ is jointly ergodic if

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^{a_1(n)}x)f_2(T_2^{a_2(n)}x)\cdots f_d(T_d^{a_d(n)}x)$$

converges (in $L^2(\mu)$) to

$$\int f_1 d\mu \cdot \int f_2 d\mu \cdots \int f_d d\mu.$$

for any $f_1, \ldots, f_d \in L^\infty(\mu)$.

Some results: single transformation

Furstenberg (1977): If T is weakly mixing [1], then for any $d \in \mathbb{N}$,

$$(T^n, T^{2n}, \dots, T^{dn})_{n \in \mathbb{N}} \text{ is jointly ergodic.}$$

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Bergelson (1987): If T is weakly mixing, and $p_1, p_2, \ldots, p_d$ are polynomials such that

- p_i is non constant, $i = 1, \ldots, d$.
- $p_i - p_j$ is non constant for $i \neq j$. Then

$$(T^{p_1(n)}, T^{p_2(n)}, \ldots, T^{p_d(n)})_{n\in\mathbb{N}} \text{ is jointly ergodic.}$$

e.g.: $(T^{n^2+n}, T^{n^2})_{n\in\mathbb{N}}$ is jointly ergodic.

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Frantzikinakis and Kra(2005): If T is totally ergodic and $p_1, \ldots, p_d$ is an *independent family of polynomials* (a linear combination along integers is non-constant), then

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Tsinas (2023): If $a_1, \ldots, a_k$ are Hardy functions of polynomial growth with linear combination far away from log, then

$$(T^{[a_1(n)]}, T^{[a_2(n)]}, \ldots, T^{[a_d(n)]})_{n\in\mathbb{N}} \text{ is jointly ergodic.}$$

(previous result in that direction by Bergelson, Moreira, Richter (2020); Tsina's talk)

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- $T_i T_j^{-1}$ is ergodic for all $i \neq j$.
- $T_1 \times T_2 \cdots \times T_d$ is ergodic for $\mu^{\otimes d} := \mu \otimes \mu \cdots \otimes \mu$.

then

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Bergelson, Leibman and Son (2016): Let $\varphi_1, \dots, \varphi_d$ be generalized linear functions. Then $(T_1^{\varphi_1(n)})_n, \dots, (T_d^{\varphi_d(n)})_n$ are jointly ergodic for μ if, and only if, both of the following conditions are satisfied:

(i) $\left(T_i^{\varphi_i(n)} T_j^{-\varphi_j(n)}\right)_n$ is ergodic for μ for all $1 \leqslant i, j \leqslant d$, $i \neq j$;
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Chu, Frantzikinakis and Host (2011) Case of polynomials of distinct growth. Recent progress in the polynomial case (Wenbo's talk) D., Ferre-Moragues, Koutsogiannis, Sun, Frantzikinakis, Kuca

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Hardy function of polynomial growth with "strongly non polynomial part" that dominates log. More later...

Host-Kra seminorms

Notation: $(X, \mathcal{X}, T_1, \ldots, T_d)$ will be written as $(X, \mathbb{Z}^d)$ and for $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, $T_n = T_1^{n_1} \cdots T_d^{n_d}$.
As in Wenbo's talk, for subgroups $G_1, \ldots, G_k$ of $\mathbb{Z}^d$, we can construct seminorms

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$|\!|\!| f |\!|\!|_{(G_1,G_2)^{\times 3}} = |\!|\!| f |\!|\!|_{G_1,G_2,G_1,G_2,G_1,G_2}$, etc.

Ingredients

Frantikinakis (2023); Best and Ferre-Moragues (2022)

$(a_1(n), \ldots, a_d(n))$ is jointly ergodic iff

1. it is *good for seminorm estimates:* for the system $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ if there exists $M \in \mathbb{N}$ such that if $f_1, \ldots, f_k \in L^\infty(\mu)$ and $\|\!| f_\ell |\!\|_{(\mathbb{Z}^d)^{\times M}} = 0$ then

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{n\in[N]^d} \prod_{i=1}^{k} T_{a_i(n)} f_i = 0,$$

2. *good for equidistribution for the system for the system* $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$, if for every $\alpha_1, \ldots, \alpha_k \in \operatorname{Spec}\left((T_n)_{n\in\mathbb{Z}^d}\right)$, not all of them being trivial, we have

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{n\in[N]^d} \exp(\alpha_1(a_1(n)) + \cdots + \alpha_k(a_k(n))) = 0$$

where $\exp(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$, and

$\operatorname{Spec}\left((T_n)_{n\in\mathbb{Z}^d}\right) = \{\alpha \in \operatorname{Hom}(\mathbb{Z}^d, \mathbb{T}) \colon\ T_n f = \exp(\alpha(n)) f,\ n \in \mathbb{Z}^d, \text{ for } f \neq 0 \in L^2(\mu)\}.$

Using Frantzikinakis criteria for joint ergodicity, our main goal is to obtain upper bounds for the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^{[a(n)]}x) f_2(T_2^{[a(n)]}x) \cdots f_d(T_d^{[a(n)]}x)$$

in terms of (box) Host-Kra seminorms.

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There exists $D \in \mathbb{N}$ such that if $\|\!| f_1 |\!\|_{(T_1, T_1T_2^{-1}, \ldots, T_1T_d^{-1})^{\times D}} = 0$, then

$$\limsup_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T_1^{[a(n)]}f_1 \cdot \ldots \cdot T_d^{[a(n)]}f_d \right\|_2 = 0. \qquad (1)$$

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...and then proceed as in Wenbo's talk...

Strategy for obtaining seminorm estimates

Steps (roughly):

1. Transform the problem into one about variable polynomials.
2. Use appropriate PET induction to find upper bounds in terms of averages of (box) seminorms.
3. Track the coefficients and use concatenation to obtain box Host-Kra seminorms. By ergodicity assumptions these are Host-Kra seminorms.

1: Reducing to variable polynomials

A sequence of real *variable polynomials* is a sequence of the form $(p_N(n))_{N,n} \subseteq \mathbb{R}$, where we assume that while the polynomials p_N might depend on N, their degrees do not.

$$p_{N,1}(n) = \frac{n^{17}}{\sqrt{N}}$$

$$p_{N,1}(n) = \Big(\frac{\sqrt{2}}{N^{e/\pi}} + \frac{N}{3}\Big)n^7 - \frac{33}{\log N}n + 1, \quad N, n \in \mathbb{N}.$$

1: Reducing to variable polynomials

Let a be a function and $(p_N)_N$ be a sequence of functions such that there exists L a positive function with $1 \prec L(x) \prec x$.

$$a(N+r) = p_N(r) + e_{N,r}, \quad \text{with} \quad e_{N,r} \ll 1.$$

for $0 \leqslant r \leqslant L(N)$, and let $f_1 \in L^\infty(\mu)$.

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for $0 \leqslant r \leqslant L(N)$, and let $f_1 \in L^\infty(\mu)$. If

$$\limsup_{N\to\infty} \sup_{|c_n|\leqslant 1} \sup_{\|f_2\|_\infty,\dots,\|f_d\|_\infty\leqslant 1} \left\| \mathbb{E}_{0\leqslant n\leqslant L(N)} c_n \prod_{i=1}^{d} T_i^{[p_N(n)]} f_i \right\|_2 = 0,$$

then we have

$$\limsup_{N\to\infty} \left\| \mathbb{E}_{1\leqslant n\leqslant N} \prod_{i=1}^{d} T_i^{[a(n)]} f_i \right\|_2 = 0$$

for all $f_2, \dots, f_d \in L^\infty(\mu)$.

Dealing with variable polynomials

How to deal with

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We need PET-induction for variable polynomials, with a tracking coefficient method. I will avoid giving details here.

After running PET induction enough times one gets a bound

$$\overline{\mathbb{E}}_{\mathsf{h}\in\mathbb{Z}^r} \left\| f_1 \right\|_{T_{c_1(\mathsf{h})}, T_{c_1(\mathsf{h})}, T_{c_1(\mathsf{h})-c_2(\mathsf{h})}, \ldots, T_{c_1(\mathsf{h})-c_\ell(\mathsf{h})}} = 0$$

Concatenation theorems (Tao and Ziegler) allows us to bound this by

$$\left\| f_1 \right\|_{(T_1, T_1 T_2^{-1}, \ldots, T_1 T_d^{-1})^{\times D}}$$

which under ergodicity assumptions equals

$$\left\| f_1 \right\|_{(\mathbb{Z}^d)^{\times D}}$$

Conditions/examples of good sequences

Let h be a Hardy field function of polynomial growth, that is there exists k and C with $h(x) \leqslant Cx^k$ for all large enough x.
Any such h can be written as

$$h(x) = s_h(x) + p_h(x) + e_h(x), \quad \text{(Richter)}$$

where s_h is a strongly non-polynomial Hardy field function,that is, there exists i with $\lim x^i / s_h(x) = \lim s_h(x)/x^{i+1} = 0$.

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Our result applies when $\log(x)/s_h(x) \to 0$. We conjecture that this can be improved to $\log(x)/h(x) \to 0$.
Our result cover cases of Hardy+tempered, for instance $x^{17} + x^{1/2}(2 + \cos\sqrt{\log x})$ or $x^{\pi}/\log x + x^{1/2}(2 + \cos\sqrt{\log x})$.

Questions / comments

1. Commuting transformations and different iterates (in progress with B. Kuca, A. Koutsogiannis, W. Sun, K. Tsinas).

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{[a_1(n)]} f_1 \cdot \ldots \cdot T_d^{[a_d(n)]} f_d.$$

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2. Non commutative setting. Recent results by Bergelson-Son, Frantzikinakis-Host.

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2. Non commutative setting. Recent results by Bergelson-Son, Frantzikinakis-Host.
3. How about topological joint ergodicity (as in Xiangdong's talk).

Dziękuję!