

The Ergodic Hierarchy of Mixing, van der Corput's difference theorem, and the ergodic theory of noncommuting operators

Nilpotent structures in topological dynamics, ergodic theory and combinatorics at Pałac Będlewo

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- 1 Review
 - Van der Corput's difference theorem and some applications
 - Mixing properties
- 2 Mixing van der Corput difference theorems
- 3 Ergodic theorems for noncommuting operators
 - Background
 - New results from mixing vdCs
- 4 Examples of systems with singular spectrum

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is **uniformly distributed** if for any open interval $(a, b) \subseteq [0, 1]$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a, b)\}| = b - a. \quad (1)$$

Theorem (van der Corput, [26])

If $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Theorem (HvdCDT1, [3, Theorem 1.4])

If \mathcal{H} is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle = 0, \quad (2)$$

for every $h \in \mathbb{N}$, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (3)$$

Theorem (HvdCDT2, [3, Page 3])

If \mathcal{H} is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h \rightarrow \infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \quad (4)$$

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (5)$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3, [3, Theorem 1.5], or [18, Lemmas 4.9, 7.5])

If \mathcal{H} is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \quad (6)$$

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (7)$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) (X, \mathcal{B}, μ, T) , and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \quad (8)$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy)

For any m.p.s. (X, \mathcal{B}, μ, T) , and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \quad (9)$$

Furstenberg's proof in [17, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^\infty$. See also [4, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg, [17])

For any m.p.s. (X, \mathcal{B}, μ, T) , any $A \in \mathcal{B}$ with $\mu(A) > 0$, and any $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-\ell n}A) > 0. \quad (10)$$

The proof presented in [9] uses HvdCT3 as Theorem 7.11, and the proof in [18] uses a variation.

Theorem (Bergelson and Leibman, [6])

For any m.p.s. $(X, \mathcal{B}, \mu, \{T_i\}_{i=1}^{\ell})$ with the T_i s commuting, any $A \in \mathcal{B}$ with $\mu(A) > 0$, and any $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$, there exists $n \in \mathbb{N}$ for which

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \dots \cap T_{\ell}^{-p_{\ell}(n)}A\right) > 0. \quad (11)$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

Some of the Ergodic Hierarchy of Mixing

Definition

Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be a m.p.s. If for every $f, g \in L_0^2(X, \mu)$

① $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle U_T^n f, g \rangle = 0$, then \mathcal{X} is **ergodic**.

② $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\langle U_T^n f, g \rangle| = 0$, then \mathcal{X} is **weakly mixing**,

③ $\lim_{n \rightarrow \infty} \langle U_T^n f, g \rangle = 0$, then \mathcal{X} is **strongly mixing**,

④ and if $L_0^2(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m \in \mathbb{Z}}$, then \mathcal{X} has **Lebesgue spectrum**.

⑤ which is the same as $(\langle U_T^n f, g \rangle)_{n=1}^\infty$ being Fourier coefficients of some $h \in L^1([0, 1], \mathcal{L})$, where \mathcal{L} is the Lebesgue measure.

These definitions also apply to individual elements $f \in L_0^2(X, \mu)$.

The Symmetric Ergodic Hierarchy of Mixing

Theorem

Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be a m.p.s. If for every $f \in L_0^2(X, \mu)$

- 1 $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle U_T^n f, f \rangle = 0$, then \mathcal{X} is *ergodic*,
- 2 $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\langle U_T^n f, f \rangle| = 0$, then \mathcal{X} is *weakly mixing*,
- 3 $\lim_{n \rightarrow \infty} \langle U_T^n f, f \rangle = 0$, then \mathcal{X} is *strongly mixing*,
- 4 \mathcal{X} has *Lebesgue spectrum* if $(\langle U_T^n f, f \rangle)_{n=1}^\infty$ are the Fourier coefficients of some $h \in L^1([0, 1], \mathcal{L})$ taking nonnegative real values.

This theorem also applies to individual elements $f \in L_0^2(X, \mu)$.

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A weak mixing van der Corput difference theorem

Theorem (MvdCT3)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \quad (12)$$

then $(x_n)_{n=1}^{\infty}$ is a *nearly weakly mixing sequence*. This means that for any other bounded sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ we morally (*but not literally*) have that

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, y_n \rangle \right| = 0. \quad (13)$$

Loosely speaking, this can be interpreted as a *weak mixing* in any ultrapower \mathcal{H} of \mathcal{H} with respect to a unitary operator induced by the left shift. Note that elements of \mathcal{H} are sequences in \mathcal{H} .

A Lebesgue spectrum vdCdt

Theorem (MvdCT1)

If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying for all $h \in \mathbb{N}$

$$\sum_{h=1}^{\infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \quad (14)$$

then $(x_n)_{n=1}^\infty$ is a *spectrally Lebesgue sequence*. In particular, if $\mathcal{H} = L^2(X, \mu)$ and $(y_n)_{n=1}^\infty \subseteq L^\infty(X, \mu)$ is *spectrally singular*, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n y_n \right\| = 0. \quad (\#)$$

Upgrading from weak convergence to the strong convergence in $\#$ necessitates a new proof of the classical vdCDT. See [10, Chapter 2] for variations of MvdCT related to other levels of mixing, as well as uniform distribution. See also [25].

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Failure of noncommutative ergodic theorems

Theorem ([15, Lemma 4.1])

Let $a, b : \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ be injective sequences and F be any subset of \mathbb{N} . Then there exist a probability space (X, \mathcal{B}, μ) , measure preserving automorphisms $T, S : X \rightarrow X$, both of them Bernoulli, and $A \in \mathcal{B}$, such that

$$\mu(T^{-a(n)}A \cap S^{-b(n)}A) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \quad (15)$$

See also [18, Page 40], [2, Example 7.1], and [7].

Noncommutative ergodic theorems 1/2

Theorem ([12, Corollary 1.7])

Let $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t \rightarrow \infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t \rightarrow \infty} \frac{t^{d+1}}{a(t)} = \infty. \quad (\text{e.g. } a(t) = t^{1.5}) \quad (16)$$

Furthermore, let (X, \mathcal{B}, μ) be a probability space and $T, S : X \rightarrow X$ be measure preserving transformations. Suppose that the system (X, \mathcal{B}, μ, T) has **zero entropy**. Then

(i) For every $f, g \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{[a(n)]} g = \mathbb{E}[f | \mathcal{I}_T] \cdot \mathbb{E}[g | \mathcal{I}_S], \quad (17)$$

where the limit is taken in $L^2(X, \mu)$.

Theorem (Continued)

(ii) For every $A \in \mathcal{B}$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A) \geq \mu(A)^3. \quad (18)$$

In [13] a similar theorem is proven for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2. In [23] it was shown that every T with **singular spectrum** must also have **zero entropy**. Note that the Horocycle flow has **zero entropy** [19] and **Lebesgue spectrum** [22]

Theorem (F., 2022)

Let (X, \mathcal{B}, μ) be a probability space and let $T, S : X \rightarrow X$ be measure preserving automorphisms for which T has *singular spectrum*. Let $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^\infty$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

(i) For any $f, g \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{k_n} g = \mathbb{E}[f | \mathcal{I}_T] \mathbb{E}[g | \mathcal{I}_S], \quad (19)$$

with convergence taking place in $L^2(X, \mu)$.

Theorem (Continued)

(ii) If $A \in \mathcal{B}$ then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A \cap S^{-k_n}A) \geq \mu(A)^3. \quad (20)$$

(iii) If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is *totally ergodic*.

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2.

Application 2/4 (A special case)

Theorem (F., 2022)

Let (X, \mathcal{B}, μ) be a probability space and $T, S : X \rightarrow X$ be measure preserving automorphisms. Suppose that T has **singular spectrum** and S is **totally ergodic**. Let $p_1, \dots, p_K \in \mathbb{Q}[x]$ be integer polynomials for which $\deg(p_1) \geq 2$ and $\deg(p_i) \geq 2 + \deg(p_{i-1})$. For any $f, g_1, \dots, g_K \in L^\infty(X, \mu)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \prod_{i=1}^K S^{p_i(n)} g_i = \mathbb{E}[f | \mathcal{I}_T] \prod_{i=1}^K \int_X g_i d\mu, \quad (21)$$

with convergence taking place in $L^2(X, \mu)$.

Application 3/4 (A special case)

Theorem (F., 2022)

Let (X, \mathcal{B}, μ) be a probability space and $T, R, S : X \rightarrow X$ be measure preserving automorphisms. Suppose that T has **singular spectrum**, R and S commute, and S is **weakly mixing**. Let $\ell \in \mathbb{N}$ and let $p_1, \dots, p_\ell \in \mathbb{Q}[x]$ be pairwise essentially distinct integer polynomials, each having degree at least 2. For any $f, h, g_1, \dots, g_\ell \in L^\infty(X, \mu)$ satisfying $\int_X g_j d\mu = 0$ for some $1 \leq j \leq \ell$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot R^n h \cdot \prod_{j=1}^{\ell} S^{p_j(n)} g_j = 0, \quad (22)$$

with convergence taking place in $L^2(X, \mu)$.

An example to justify our assumptions

Consider the m.p.s. $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, T, S)$ with $S(x, y) = (x + 2\alpha, y + x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x, y) = (x, y + x)$. We see that $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, S)$ and $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, T)$ are both **zero entropy** systems that are not **weakly mixing**, and the former is **totally ergodic**. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x, y) = e^{2\pi i(x-y)}$, $f_1(x, y) = e^{2\pi iy}$, and $f_2(x, y) = e^{-2\pi ix}$, we see that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f_0(x, y) S^n f_1(x, y) S^{\frac{1}{2}(n^2-n)} f_2(x, y) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i((1-n)x - y + y + nx + (n^2-n)\alpha - x - (n^2-n)\alpha)} = 1 \neq 0. \end{aligned}$$

Sets of K but not $K + 1$ recurrence?

Theorem ([14, Theorem 1.4 and Corollary 4.4])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.

- (i) If (X, \mathcal{B}, μ) is a probability space and $S_1, S_2, \dots, S_{k-1} : X \rightarrow X$ are commuting measure preserving transformations, then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which

$$\mu(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \dots \cap S_{k-1}^{-n}A) > 0. \quad (23)$$

- (ii) There exists a m.p.s. (X, \mathcal{B}, μ, T) and a set $A \in \mathcal{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-kn}A) = 0. \quad (24)$$

Application 4/4

Theorem (F., 2022)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let (X, \mathcal{B}, μ) be a probability space and $S_1, S_2, \dots, S_{k-1} : X \rightarrow X$ commuting measure preserving automorphisms. Let $T : X \rightarrow X$ be an measure preserving automorphism with *singular spectrum*, and for which $\{T, S_1, S_2, \dots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \dots \cap S_{k-1}^{-n}A) > 0. \quad (25)$$

Since the system $(\mathbb{T}^2, \mathcal{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \delta_\alpha$.

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Examples of systems with singular spectrum

In [5, Proposition 2.9] it is shown that if (X, \mathcal{B}, μ) is a standard probability space, and $\text{Aut}(X, \mathcal{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are **weakly mixing** and rigid is a generic set. Since any rigid automorphism has **singular spectrum**, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \text{Aut}(X, \mathcal{B}, \mu)$ denote the collection of **strongly mixing** transformation, and note that \mathcal{S} is a meager set since an automorphism cannot simultaneously be rigid and **strongly mixing**. Since \mathcal{S} is not a complete metric space with respect to the topology induced by the strong operator topology, a new topology was introduced in [24], with respect to which \mathcal{S} is a complete metric space. It is shown in the Corollary to Theorem 7 of [24] that a generic $T \in \mathcal{S}$ has **singular spectrum**, and such a T is mixing of all orders due a well known result of Host [20]. See [11] and [21] for concrete examples of $T \in \mathcal{S}$ that have **singular spectrum**. See also [1], [8], and [16].

- [1] C. Aistleitner and M. Hofer.
On the maximal spectral type of a class of rank one transformations.
Dyn. Syst., 27(4):515–523, 2012.
- [2] D. Berend.
Joint ergodicity and mixing.
J. Analyse Math., 45:255–284, 1985.
- [3] V. Bergelson.
Weakly mixing PET.
Ergodic Theory Dynam. Systems, 7(3):337–349, 1987.

- [4] V. Bergelson.
Ergodic Ramsey theory—an update.
In *Ergodic theory of Z^d actions (Warwick, 1993–1994)*,
volume 228 of *London Math. Soc. Lecture Note Ser.*, pages
1–61. Cambridge Univ. Press, Cambridge, 1996.
- [5] V. Bergelson, A. del Junco, M. Lemańczyk, and
J. Rosenblatt.
Rigidity and non-recurrence along sequences.
Ergodic Theory Dynam. Systems, 34(5):1464–1502, 2014.
- [6] V. Bergelson and A. Leibman.
Polynomial extensions of van der Waerden’s and Szemerédi’s
theorems.
J. Amer. Math. Soc., 9(3):725–753, 1996.

- [7] V. Bergelson and A. Leibman.
Failure of the Roth theorem for solvable groups of exponential growth.
Ergod. Th. and Dynam. Sys., 24:45–53, 2004.
- [8] A. I. Bufetov and B. Solomyak.
On substitution automorphisms with pure singular spectrum.
Math. Z., 301(2):1315–1331, 2022.
- [9] M. Einsiedler and T. Ward.
Ergodic theory with a view towards number theory, volume 259 of *Graduate Texts in Mathematics*.
Springer-Verlag London, Ltd., London, 2011.

References IV

- [10] S. Farhangi.
Topics in ergodic theory and ramsey theory.
PhD dissertation, the Ohio State University, 2022.
- [11] B. Fayad.
Smooth mixing flows with purely singular spectra.
Duke Math. J., 132(2):371–391, 2006.
- [12] N. Frantzikinakis.
Furstenberg systems of Hardy field sequences and applications.
J. Anal. Math., 147(1):333–372, 2022.
- [13] N. Frantzikinakis and B. Host.
Multiple recurrence and convergence without commutativity.
arXiv:2111.01518v2 [math.DS], 2021.

- [14] N. Frantzikinakis, E. Lesigne, and M. Wierdl.
Sets of k -recurrence but not $(k + 1)$ -recurrence.
Ann. Inst. Fourier (Grenoble), 56(4):839–849, 2006.
- [15] N. Frantzikinakis, E. Lesigne, and M. Wierdl.
Random sequences and pointwise convergence of multiple ergodic averages.
Indiana Univ. Math. J., 61(2):585–617, 2012.
- [16] K. Frączek and M. Lemańczyk.
On mild mixing of special flows over irrational rotations under piecewise smooth functions.
Ergodic Theory Dynam. Systems, 26(3):719–738, 2006.

- [17] H. Furstenberg.
Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions.
J. Analyse Math., 31:204–256, 1977.
- [18] H. Furstenberg.
Recurrence in ergodic theory and combinatorial number theory.
Princeton University Press, Princeton, N.J., 1981.
M. B. Porter Lectures.
- [19] B. M. Gurevič.
The entropy of horocycle flows.
Dokl. Akad. Nauk SSSR, 136:768–770, 1961.

- [20] B. Host.
Mixing of all orders and pairwise independent joinings of systems with singular spectrum.
Israel J. Math., 76(3):289–298, 1991.
- [21] I. Klemes and K. Reinhold.
Rank one transformations with singular spectral type.
Israel J. Math., 98:1–14, 1997.
- [22] O. S. Parasyuk.
Flows of horocycles on surfaces of constant negative curvature.
Uspehi Matem. Nauk (N.S.), 8(3(55)):125–126, 1953.

- [23] V. A. Rohlin.
Lectures on the entropy theory of transformations with invariant measure.
Uspehi Mat. Nauk, 22(5 (137)):3–56, 1967.
- [24] S. V. Tikhonov.
A complete metric on the set of mixing transformations.
Mat. Sb., 198(4):135–158, 2007.
- [25] A. Tserunyan.
A Ramsey theorem on semigroups and a general van der Corput lemma.
J. Symb. Log., 81(2):718–741, 2016.

- [26] J. G. van der Corput.
Diophantische Ungleichungen. I. Zur Gleichverteilung Modulo
Eins.
Acta Math., 56(1):373–456, 1931.