Asymptotic pairs in actions of amenable groups

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Based on joint work with Mateusz Więcek

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Thus zero entropy systems can be characterized precisely as **factors of NAP systems.**

Following the global tendency to generalize everything possible to a wider class of actions, I asked Mateusz the following question:

- Is there a reasonable definition of an asymptotic pair in an action of a countable amenable group G, so that an analogous characterization of zero entropy G-action holds?

Comments

There is a very simple notion of an asymptotic pair:

- A pair $x \neq y$ is "asymptotic" if for any $\varepsilon > 0$ we have $d(gx, gy) < \varepsilon$ for all but finitely many $g \in G$.

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But we want to work with general (not necessarily orderable) groups. Here, the first question is: how can one define the "future" of an orbit?

Mutiorders

Let G be a group. An *invariant random order* (IRO) is a family of total orders $\mathcal{O}$ on G and a probability measure ν on $\mathcal{O}$ invariant under the G-action given by

$$a\, g(\prec)\, b \iff ga \prec gb \ \ (\prec \in \mathcal{O}).$$

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Multiorders were introduced by D., Oprocha, Więcek and Zhang (2022) and we proved that they exist on any countable amenable group.

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A multiordered G-action will be denoted by (X, μ, G, φ), where $\varphi : X \to \mathcal{O}$ is the factor map to a multiorder.

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Note that every $\mathbb{Z}$-action is multiordered by the one-point factor $\{<\}$.

Asymptotic pairs in multiordered systems

In a multiordered system, the "future" of each orbit is easily defined as the set $\{n^{\prec}x : n \geq 1\}$ and "remote future" is the set $\{n^{\prec}x : n \geq N\}$. This allows one to define asymptotic pairs, as follows:

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Note that for $\mathbb{Z}$-actions (X, μ, T) and φ defined by $\varphi(x) = <$ for all $x \in X$, this definition generalizes the standard notion of an asymptotic pair.

Multiordered topological G-actions

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Instead, we will introduce a "topological multiorder" using the theory of tilings. Such a multiorder will arise from a genuine (i.e., compact) topological system in which it is large in both topological and measure-theoretic sense (although still without being compact).

Tiling systems of countable amenable groups

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Definition

A *tiling* $\mathcal{T}$ of a countable group G is a partition of G into finite sets T (called tiles) such that there exists a finite family $\mathcal{S}$ of sets $S \subset G$ (called *shapes*, each containing the unit e, such that for every tile T there exists a shape $S \in \mathcal{S}$ and a *center* $c \in G$ such that $T = Sc$. A tiling can be encoded symbolically, by placing at the centers of tiles symbols assigned bijectively to the shapes.

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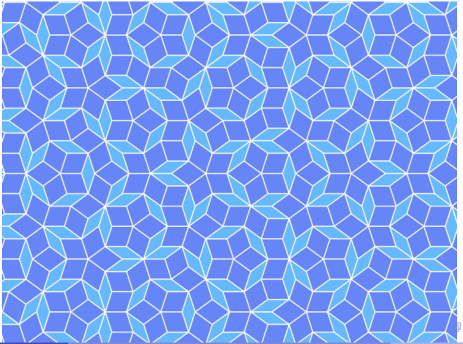

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Definition

A *dynamical tiling* T of a countable amenable group G is a G-action on a closed invariant family of tilings with a common family of shapes $\mathcal{S}$.

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Summary of notation:

$\mathcal{T}$ - (static) tiling,

T - tile of $\mathcal{T}$,

T - dynamical tiling,

$\mathbf{T} = \bigvee_{k\geq 1} \mathsf{T}_k$ - tiling system (topological joining of dynamical tilings),

$\boldsymbol{\mathcal{T}} = (\mathcal{T}_k)_{k\geq 1}$ - element of the tiling system (sequence of static tilings).

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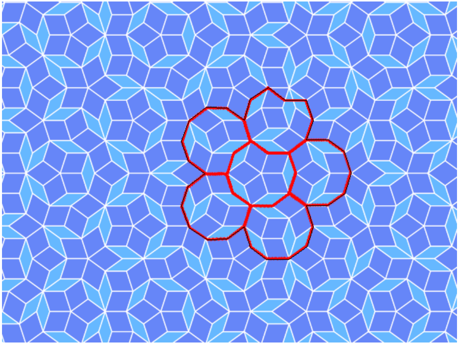

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2. for $k \geq 1$ we know that each $S \in \mathcal{S}_{k+1}$ splits (in a unique way - this is due to determinism) as a union of tiles of order k. We demand that the subtiles of S are ordered linearly (from 1 to the number of subtiles).

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We are interested in these elements $\mathcal{T} \in \mathbf{T}$ which determine an order of type $\mathbb{Z}$ on G:

Ordered tiling systems

Definition

An element $\mathcal{T} \in \mathbf{T}$ is *straight* if

1. The union (over $k \geq 1$) of the central tiles of $\mathcal{T}_k$ (the tiles of $\mathcal{T}_k$ containing e) covers G,
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Theorem (D., Oprocha, Więcek, Zhang, 2022)

If $\mathbf{T}$ is an ordered tiling system then the collection $\mathbf{T}_{\text{STR}}$ of all straight elements $\mathcal{T} \in \mathbf{T}$ is residual and has full invariant measures in $\mathbf{T}$.

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If $\mathcal{O}_{\mathbf{T}}$ is a topological multiorder then the mapping $\mathcal{T} \mapsto \prec_{\mathcal{T}}$ is defined and continuous on $\mathbf{T}_{\mathrm{STR}}$, and it commutes with the action of G (the action on $\mathbf{T}$ is the shift, while on $\mathcal{O}_{\mathbf{T}}$ the action was defined earlier by $a\, g(\prec)\, b \iff ga \prec gb$).

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For each invariant measure μ on $\mathbf{T}$, this mapping is a measure-theoretic factor map from $(\mathbf{T}_{\mathsf{STR}}, \mu, G)$ to the multiorder $(\mathcal{O}_{\mathbf{T}}, \nu, G)$, where ν is the image of μ via this map.

Topologically multiordered systems (finally!)

Theorem (D., Oprocha, Więcek, Zhang, 2022)

Every topological multiorder $\mathcal{O}_{\mathbf{T}}$ has the *uniform Følner property*: For each $\varepsilon > 0$ and each finite set $K \subset G$, there exists n such that for any $\prec_{\mathcal{T}} \in \mathcal{O}_{\mathbf{T}}$, any order interval of length at least n is (K, ε)-invariant.

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Definition (D., Więcek, 2023)

A topological G-action (X, G) is *topologically multiordered* if there exists an ordered tiling system $\mathbf{T}$ and a topological factor map $\pi : X \to \mathbf{T}$ such that the preimage $X_{\mathsf{STR}} = \pi^{-1}(\mathbf{T}_{\mathsf{STR}})$ is dense in X.

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If (X, G) is topologically multiordered then on X_{STR} we define the map φ onto $\mathcal{O}_{\mathcal{T}}$ by

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The mapping φ is defined and continuous on a residual set of full invariant measure.

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We denote such a topologically multiordered system by (X, G, π, φ).

Main theorems

Theorem (D., Więcek, 2023)

1. Let (X, μ, G, φ) be a multiordered measure-preserving action of a countable amenable group. Suppose that $h_\mu(X|\varphi) > 0$. Then there exists a φ-asymptotic pair in X. Moreover, the union of such pairs has positive measure μ.

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The passage from (1) to (2) is via picking a measure μ on X of positive entropy, looking at the direct product of (X, μ, G) with $(\mathcal{O}, \nu, G)$ and letting φ be the projection on the second coordinate. Then $h_\mu(X|\varphi) = h_\mu(X) > 0$ and (1) applies. Almost all $\prec$ are obtained for ergodic components of ν by invariance of the set of $\prec$ for which a pair exists.

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Statement (1) cannot be deduced from (2), because the graph of φ may have measure zero in the product.

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We call it *odometric tiling system*, becuse it shares some common features with usual odometers. This tiling system also has entropy zero.

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We now extend our initial system (X, G) by its direct product with an odometric tiling system of entropy zero. This extension is zero-dimensional, so we can represent it symbolically by *symbolic arrays*. The centers of the tiles are marked in respective rows by some special markers (for example by stars). This is a topologically multiordered extension.

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It is an extremely easy exercise that in such an inverse limit there are no asymptotic pairs.

THANK YOU!