Modern tools in jointly ergodic problems

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Multiple ergodic averages

Theorem (Mean ergodic theorem)

Let (X, μ, T) be a measure preserving system. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n f = \int_X f\, d\mu \text{ for all } f \in L^\infty(\mu) \Leftrightarrow \; T \text{ is ergodic for } \mu.$$

Note: all limits in this talk are taken in $L^2(\mu)$.

We know that the following L^2-limits exists

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n f_1 \cdot \ldots \cdot T^{dn} f_d$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d \; (T_i \text{ is m.p.s. and } T_i T_j = T_j T_i)$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} (\prod_{i=1}^{d} T_i^{p_{1,i}(n)}) f_1 \cdot \ldots \cdot (\prod_{i=1}^{d} T_i^{p_{d,i}(n)}) f_d \; (p_{i,j} \text{ are polynomials})$$

Jointly ergodicity problem

Question (Jointly ergodicity problem)

When is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d = \left(\int_X f_1\, d\mu\right) \cdot \ldots \cdot \left(\int_X f_d\, d\mu\right) \text{ for all } f_1, \ldots, f_d \in L^\infty(\mu)?$$

Alternative statement: when is $T_1 \times \cdots \times T_d$ ergodic for the diagonal measure μ_d^Δ on X^d?

Theorem (Berend and Bergelson (1984))

The answer is yes if and only if

- $T_i T_j^{-1}$ is ergodic for μ for all $i \neq j$;
- $T_1 \times \cdots \times T_d$ is ergodic for the product measure μ^d on X^d.

More general questions

We say that $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ is a $\mathbb{Z}^d$-system if $(X, \mathcal{B}, \mu)$ is a probability space and $T_n\colon X \to X$ are measure-preserving transformations with $T_{\mathbf{0}} = id$ and $T_m \circ T_n = T_{m+n}$ for all $m, n \in \mathbb{Z}^d$.
Convention: write $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}, \mu, T_1, \dots, T_d)$ for $\mathbb{Z}$- and $\mathbb{Z}^d-$ systems.

Let $p_1, \dots, p_k\colon \mathbb{Z} \to \mathbb{Z}^d$ be functions. We say that $(p_1(n), \dots, p_k(n))$ is jointly ergodic for μ if for all $f_1, \dots, f_k \in L^\infty(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_{p_1(n)} f_1 \cdot \ldots \cdot T_{p_k(n)} f_k = \left(\int_X f_1 \, d\mu\right) \cdot \ldots \cdot \left(\int_X f_k \, d\mu\right).$$

Question

When is $(p_1(n), \dots, p_k(n))$ is jointly ergodic?

Statement of the conjecture

Let $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system. We say that $(a_n)_{n\in\mathbb{Z}}, a_n \in \mathbb{Z}^d$ is an ergodic sequence for μ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_{a_n} f = \int_X f \, d\mu \text{ for all } f \in L^\infty(\mu).$$

Example: In a $\mathbb{Z}$-system $(X, \mathcal{B}, \mu, T)$, T is ergodic $\Leftrightarrow$ $\mathbb{N}$ is an ergodic sequence for μ.

Conjecture (Joint Ergodicity Conjecture)

Let $p_1, \ldots, p_k : \mathbb{Z} \to \mathbb{Z}^d$ be polynomials. $(p_1(n), \ldots, p_k(n))$ is jointly ergodic for μ if and only if

- $(T_{p_i(n)-p_j(n)})_{n\in\mathbb{Z}}$ is an ergodic sequence for μ for all $i \neq j$;
- $(T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}}$ is an ergodic sequence for μ^k.

Joint Ergodicity Conjecture

Conjecture (Joint Ergodicity Conjecture)

$(p_1(n), \dots, p_k(n))$ is jointly ergodic for μ if and only if

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Linear iterates:

$(T_1^n, \dots, T_d^n)$: Berend and Bergelson (1984).

(T_1^n, T_2^n), $k = 2$ and T_1, T_2 form a nilpotent group: Bergelson and Leibman (2002).

$(T_1^{[\alpha_1 n]}, \dots, T_d^{[\alpha_d n]})$: Bergelson, Leibman and Son (2016).

Iterates of distinct degrees:

$(T_1^{p_1(n)}, \dots, T_d^{p_d(n)})$ and $p_1, \dots, p_d$ are of distinct degrees: Chu, Frantzikinakis and Host (2011).

Any more cases?

Main result

Theorem (Donoso, Koutsogiannis and S.)

The Jointly Ergodicity Conjecture holds for $(T_1^{p(n)}, \ldots, T_d^{p(n)})$ for any polynomial p.

Further developments:

Donoso, Ferre, Koutsogiannis and S.: a slightly more general case.

Frantzikinakis and Kuca: $(T_1^{p_1(n)}, \ldots, T_d^{p_d(n)})$ with p_i being polynomials.

Donoso, Koutsogiannis and S.: $(T_1^{p(n)}, \ldots, T_d^{p(n)})$ with p being a Hardy field function.

A reproof of the linear case, the "only if" part

Theorem (Berend and Bergelson)

$(T_1^n, \ldots, T_d^n)$ is jointly ergodic for μ if and only if

- $((T_i T_j^{-1})^n)_{n\in\mathbb{Z}}$ is an ergodic sequence for μ for all $i \neq j$;
- $(T_1^n \times \cdots \times T_d^n)_{n\in\mathbb{Z}}$ is an ergodic sequence for μ^d.

The "only if" direction:

If $T_1^n f = T_2^n f$ for some non-constant f, then set $f_1 = f_2 = f$, $f_3 = \cdots = f_d = 1$.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d = \int_X \mathbb{E}(f|I(T_1))^2 d\mu \neq (\int_X f\, d\mu)^2$$

If $(T_1 \times \cdots \times T_d)^n g = g$, suppose that $g = f_1 \otimes \cdots \otimes f_d$.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d = f_1 \cdot \ldots \cdot f_d \neq (\int_X f_1\, d\mu) \cdot \ldots \cdot (\int_X f_d\, d\mu)$$

Host-Kra measures

Let $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system and $G \subseteq \mathbb{Z}^d$ be a group of measure preserving transformations on X.
$I(G)$: the σ-algebra of G-invariant sets.
Let $\mathcal{D} \subseteq \mathcal{B}$. Then $\mu \times_{\mathcal{D}} \mu$ is the relative independent product:

$$\int_{X\times X} f \otimes g \, d(\mu \times_{\mathcal{D}} \mu) := \int_X \mathbb{E}(f|\mathcal{D})\mathbb{E}(g|\mathcal{D}) \, d\mu.$$

Host-Kra measures:
Let $G_1, \ldots, G_d$ be subgroups of $\mathbb{Z}^d$.

$$\mu_{G_1} := \mu \times_{I(G_1)} \mu$$

$$\mu_{G_1,\ldots,G_d} := \mu_{G_1,\ldots,G_{d-1}} \times_{I(G_d^{\Delta})} \mu_{G_1,\ldots,G_{d-1}},$$

where $G^{\Delta} = \{(g, \ldots, g) \colon g \in G\}$.
$\mu_{G_1,\ldots,G_d}$ is a measure on X^{2^d}.

Host-Kra seminorms and factors

Host-Kra seminorms: For $f \in L^\infty(\mu)$, let

$$\|f\|_{G_1,\ldots,G_d} := \Big(\int_{X^{2^d}} f^{\otimes 2^d}\, d\mu_{G_1,\ldots,G_d}\Big)^{\frac{1}{2^d}}.$$

Theorem (Host)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d \ll \|f_1\|_{T_1, T_1 T_2^{-1}, \ldots, T_1 T_d^{-1}}.$$

Corollary

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d \ll \|f_1\|_{\mathbb{Z}^d,\ldots,\mathbb{Z}^d}$$

if all of $T_1, T_1 T_2^{-1}, \ldots, , T_1 T_d^{-1}$ are ergodic.

Theorem (Frantzikinakis; Best and Férre Moragues)

$(p_1(n), \dots, p_k(n))$ is jointly ergodic for μ if
(1) $(p_1(n), \dots, p_k(n))$ is *good for seminorm estimate*, meaning that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_d^{p_d(n)} f_d = 0 \text{ if } \|f_i\|_{\mathbb{Z}^d,\dots,\mathbb{Z}^d} = 0$$

for all $1 \le i \le d$.
(2) $(p_1(n), \dots, p_k(n))$ is *good for equidistribution*, meaning that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_d^{p_d(n)} f_d = \left(\int_X f_1 \, d\mu\right) \cdot \cdots \cdot \left(\int_X f_d \, d\mu\right)$$

for all $f_1, \dots, f_d$ which is measurable with respect to the largest rotation factor of X.

A reproof of the linear case, the "if" part

Theorem (Berend and Bergelson)

$(T_1^n, \ldots, T_d^n)$ is jointly ergodic for μ if and only if

- $((T_i T_j^{-1})^n)_{n\in\mathbb{Z}}$ is an ergodic sequence for μ for all $i \neq j$;
- $(T_1^n \times \cdots \times T_d^n)_{n\in\mathbb{Z}}$ is an ergodic sequence for μ^d.

For rotation systems, the blue condition implies good for equidistribution condition (in fact it implies jointly ergodic).

Corollary

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^n f_1 \cdot \ldots \cdot T_d^n f_d \ll \|f_1\|_{\mathbb{Z}^d,\ldots,\mathbb{Z}^d}$$

if all of $T_1, T_1 T_2^{-1}, \ldots, , T_1 T_d^{-1}$ are ergodic.

The red condition implies good for seminorm control condition.

Outline of the proof of the main result

Theorem (Donoso, Ferre, Koutsogiannis and S.)

$(T_1^{n^2}, \dots, T_d^{n^2})$ is jointly ergodic for μ if the following conditions hold:

- $((T_i T_j^{-1})^{n^2})_{n\in\mathbb{Z}}$ is an ergodic sequence for μ for all $0 \leq i < j \leq d$.
- $(T_1^{n^2} \times \cdots \times T_d^{n^2})_{n\in\mathbb{Z}}$ is an ergodic sequence for μ^d.

The only if part is relatively easy.

For rotation systems, the blue condition implies good for equidistribution condition (in fact it implies jointly ergodic).

It suffices to show that there exists some ergodic $S_1, \dots, S_k$ such that if $\|f_1\|_{S_1,\dots,S_k} = 0$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^{n^2} f_1 \cdot \ldots \cdot T_d^{n^2} f_d = 0.$$

Difficulty 1

If X is not a $\mathbb{Z}$-system, then Host's Theorem does not prove an upper bound for

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_{p_1(n)} f_1 \cdot \ldots \cdot T_{p_d(n)} f_d.$$

Example: Using Van der Corput trick, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^{n^2} f_1 \cdot T_2^{n^2} f_2 \ll \lim_{N\to\infty} \frac{1}{N^3} \sum_{h_1=0}^{N-1} \sum_{h_2=0}^{N-1} \sum_{h_3=0}^{N-1} \|f_1\|_{S_{\mathbf{h},1},\ldots,S_{\mathbf{h},7}},$$

where $\mathbf{h} = (h_1, h_2, h_3)$, $R = T_1 T_2^{-1}$

$$S_{\mathbf{h},1} = T_1^{-2h_1}, S_{\mathbf{h},2} = R^{2h_2}, S_{\mathbf{h},3} = T_1^{-2h_1} R^{2h_2}, S_{\mathbf{h},4} = R^{2h_3},$$

$$S_{\mathbf{h},5} = T_1^{-2h_1} R^{2h_3}, S_{\mathbf{h},6} = R^{2(h_2+h_3)}, S_{\mathbf{h},7} = T_1^{-2h_1} R^{2(h_2+h_3)}.$$

Concatenation theorem

Theorem (Concatenation theorem, Tao and Ziegler)

Let I be a finite set and $H_{j,i}, 1 \leq j \leq k, i \in I$ be subgroups of $\mathbb{Z}^d$. If

$$\frac{1}{|I|^2} \sum_{i,i' \in I} \|f\|_{\{H_{j,i}+H_{j',i'}\}_{1 \leq j,j' \leq k}} = 0,$$

then

$$\frac{1}{|I|} \sum_{i \in I} \|f\|_{H_{1,i},\ldots,H_{k,i}} = 0.$$

$$S_{\mathbf{h},1} = T_1^{-2h_1}, S_{\mathbf{h},2} = R^{2h_2}, S_{\mathbf{h},3} = T_1^{-2h_1} R^{2h_2}, S_{\mathbf{h},4} = R^{2h_3},$$
$$S_{\mathbf{h},5} = T_1^{-2h_1} R^{2h_3}, S_{\mathbf{h},6} = R^{2(h_2+h_3)}, S_{\mathbf{h},7} = T_1^{-2h_1} R^{2(h_2+h_3)}.$$

For example, $S_{\mathbf{h},3} + S_{\mathbf{h},5} = \langle T_1^2, R^2 \rangle = \langle T_1^2, T_2^2 \rangle$ "almost surely".

Concatenation theorem

Corollary

Let G_i be the group generated by $S_{\mathbf{h},i}$ for all $\mathbf{h} \in \mathbb{Z}^3$, then

$$\lim_{N\to\infty} \frac{1}{N^3} \sum_{h_1=0}^{N-1} \sum_{h_2=0}^{N-1} \sum_{h_3=0}^{N-1} \|f_1\|_{S_{\mathbf{h},1},\dots,S_{\mathbf{h},7}} \ll \|f_1\|_{G_1,\dots,G_1,G_2,\dots,G_2,\dots,G_7,\dots,G_7},$$

where in total there are finitely many (49 in this case) G_i on the RHS.

$$G_1 = \langle T_1^2 \rangle, G_2 = G_4 = G_6 = \langle R^2 \rangle, G_3 = G_5 = G_7 = \langle T_1^2, T_2^2 \rangle$$

So if T_1 and R are ergodic (true by the red condition), then $(T_1^{n^2}, T_2^{n^2})$ is good for seminorm control, and we are done.

Difficulty 2

Conjecture (Joint Ergodicity Conjecture)

$(p_1(n), \dots, p_k(n))$ is jointly ergodic for μ if and only if

- $(T_{p_i(n)-p_j(n)})_{n\in\mathbb{Z}}$ is an ergodic sequence for μ for all $i \neq j$;
- $(T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}}$ is an ergodic sequence for μ^d.

Recall that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_1^{n^2} f_1 \cdot T_2^{n^2} f_2 \ll \lim_{N\to\infty} \frac{1}{N^3} \sum_{h_1=0}^{N-1} \sum_{h_2=0}^{N-1} \sum_{h_3=0}^{N-1} \|f_1\|_{S_{\mathbf{h},1},\dots,S_{\mathbf{h},7}}.$$

What about $(T_1^{n^2}, \dots, T_3^{n^2})$ or $(T_1^{n^3}, T_2^{n^3})$?
Problem: the red condition relies on the groups $G_1, \dots, G_7$, which reply on $S_{\mathbf{h},1}, \dots, S_{\mathbf{h},7}$, which in general is difficulty to compute.

Solution: coefficient tracking method. Instead of computing $S_{\mathbf{h},1}, \dots, S_{\mathbf{h},7}$ directly, we discover some relations between $S_{\mathbf{h},1}, \dots, S_{\mathbf{h},7}$ which are preserved under the van der Corput trick.

Thank you for your attention