An extension of a result by Furstenberg and Glasner

Xiangdong Ye
(Joint work with Wen Huang and Song Shao)

School of Mathematics, USTC

June 6, 2023

Chapters

1. An introduction-levels

It is clear that there are following levels of "bigness" of subsets S of $\mathbb{N}$ or $\mathbb{Z}$.

1. syndetic or piecewise syndetic;
2. positive upper density or positive upper Banach density;
3. $\sum_{n \in S} \frac{1}{n} = \infty$.

Basically, one uses topological dynamics, ergodic theory and harmonic analysis (higher order) to deal with them respectively.

1. An introduction: the result

In today's talk I will show how to prove the following result using topological methods. [1]

Theorem (Huang-Shao-Y., arXiv:2301.07873)

Let $d \in \mathbb{N}$ and p_i be an integral polynomial with $p_i(0) = 0$, $1 \le i \le d$. If F is piecewise syndetic in $\mathbb{Z}$, then

$$\{(m,n) \in \mathbb{Z}^2 : m + p_1(n) \in F, \dots, m + p_d(n) \in F\}$$

is piecewise syndetic in $\mathbb{Z}^2$.

General ideas of the proof: We first transfer the question into the dynamical one, and then use the nilpotent structure (known before) and the saturation theorem for polynomials (developing here) to solve them.

[1] A polynomial P is integral if $P(\mathbb{Z}) \subset \mathbb{Z}$.

1. Piecewise syndetic

- $F \subset \mathbb{Z}$ is syndetic if there is $M > 0$ s.t. for each $x \in \mathbb{Z}$, $B_M(x) \cap F \neq \emptyset$; $F \subset \mathbb{Z}$ is piecewise syndetic if there are a syndetic set F_1 and intervals I_n with $|I_n| \longrightarrow \infty$ with
$$F \supset F_1 \cap (\cup_{n \in \mathbb{N}} I_n).$$
- $F \subset \mathbb{Z}^2$ is syndetic if there is $M > 0$ s.t. for each $x \in \mathbb{Z}^2$, $B_M(x) \cap F \neq \emptyset$; $F \subset \mathbb{Z}^2$ is piecewise syndetic if $\exists$ a syndetic set F_1 and intv. I_n, J_n with $|I_n|, |J_n| \longrightarrow \infty$ with
$$F \supset F_1 \cap (\cup_{n \in \mathbb{N}} I_n \times J_n).$$
- $\cup_{n \in \mathbb{N}} I_n$ or $\cup_{n \in \mathbb{N}} I_n \times J_n$ is called **a thick set.**

The set of all piecewise syndetic subsets of G will be denoted by $\mathcal{F}_{ps}(G)$, or simply $\mathcal{F}_{ps}$.

1. Motivations

Syndeticity appears naturally in the study of dynamical systems.

A tds (X, G) is **minimal** if each $x \in X$ the orbit $\{gx : g \in G\}$ is dense in X. It is known that if (X, G) is minimal then for each $x \in X$ and each neighborhood U of x,

$$N(x, U) = \{g \in G : gx \in U\}$$

is syndetic.

By Furstenberg's corresponding principle, a piecewise syndetic subset S is related to a minimal system by considering the indication function 1_S in $\{0, 1\}^{\mathbb{Z}}$.

1. Motivations

We remark that

★ The result was proved by Furstenberg and Glasner (1998) when $p_i(n) = in, 1 \leq i \leq d$. Later, Beiglböck (2009) provided a simple proof.

★ Bergelson-Leibman (1996) showed that if $F \in \mathcal{F}_{ps}$ then

$$\{(m,n) \in \mathbb{Z}^2 : m + p_1(n), \ldots, m + p_d(n) \in F\}$$

is infinite.

★ Our result is one of the open questions asked by Bergelson and Hindman (2001).

1. Dynamical version of the result

The dynamical version of our result is:

Theorem

Let (X, T) be minimal. Then for each $x \in X$ and each neighborhood U of x, one has

$$\{(m, n) \in \mathbb{Z}^2 : T^{m+P_1(n)}x \in U, \ldots, T^{m+P_d(n)}x \in U\}$$

is piece-wise syndetic.

Later, I will give another dynamical version of our result which is convenient to prove using dynamical methods.

1. A question of Furstenberg

The other motivation is the following.

In some survey paper in 1981, Furstenberg wrote:

"We will see in the next section that the latter property (means multiple recurrence) always holds for some point of any system (X, T).

On the other hand **we do not know** *if there always exists a point x such that $(x, x, \ldots, x)$ is a uniformly recurrent point (mean minimal point) for $T \times T^2 \times \cdots \times T^d$."*

1. Multiply minimal point

The difficulty of the problem is that we need to find a minimal system and know exactly the return times

$$N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\},$$

where U is an open neighborhood of x.

Theorem (Huang-Shao-Ye, 2021)

There is a minimal weakly mixing system which ***has no multiply minimal point****.*

In fact, for this system (X, T) and each point $x \in X$, (x, x) is $(X \times X, T \times T^2)$ recurrent, but not minimal.

1. Multiply minimal point

In fact, we also have some positive information.

Theorem (Huang-Shao-Ye, 2021. Linear-Theorem)

For any minimal system (X,T), $d \geq 2$, and any non-empty open set U, there exists $x \in U$ such that

$$\{n \in \mathbb{Z} : T^n x \in U, \ldots, T^{dn} x \in U\}$$

is piecewise syndetic.

Rmk: We conjecture that the result is sharp, i.e. one can not show that there is a dense G_δ-set X_0 such that for each $x \in X_0$ and each neighborhood U of x, the above holds.

(For a minimal PI system the property is equivalent to the existence of a multiply minimal point.)

4. The questions

Now let $P_1, \ldots, P_k$ be a finite collection of integral polynomials with $P_i(0) = 0$, we ask the following question.

Question

Let (X, T) be a minimal system and U be a non-empty open set. Is it true that there is $x \in U$ such that

$$\{n \in \mathbb{Z} : T^{P_1(n)}x \in U, \ldots, T^{P_d(n)}x \in U\} \in \mathcal{F}_{ps}?$$

The question has an affirmative answer (proved in the same paper) and also has a combinatoric counterpart. This also stimulates us to consider the double case (m, n).

2. A general consideration

Assume that we have a problem P for a minimal (ergodic) system (X, T) . If

1. we can show that each minimal (ergodic) system has a factor Z;

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \downarrow & & \downarrow \pi \\ Z & \xrightarrow[S]{} & Z \end{array}$$

2. we can reduce the problem P from (X, T) to (Z, S);
3. we can solve the problem P on the factor Z,

then we solve the original problem P.

2. Remarks

We remark:

1. It is very important to find a suitable factor Z (**structure theorems**). It can not be too "large" or too "small". It turns out that the **pro-nilfactor** (by previous results Host-Kra-Maass, Shao-Y., Glasner-Gutman-Y.) is a good candidate in certain situations.
2. For the second step, we need to understand $\pi : X \to Z$ very well. We will do it by proving **saturation theorems**.
3. For the third step, we use the nice algebraic structures of nilsystems.

2. Topological dynamics

By a topological dynamical system (for short tds) we mean a pair (X, G), where X is a compact metric space [2] and G is a topological group acting on X.

When $G = \mathbb{Z}$ we write $(X, \mathbb{Z})$ as (X, T), where $T : X \to X$ is a homeomorphism from X to X.

When $G = \mathbb{Z}^d$ for some $d \geq 2$ we write $(X, \mathbb{Z}^d)$ as $(X, \langle T_1, \ldots, T_d \rangle)$, where $T_i : X \to X$ is a homeomorphism from X to X and $T_i \circ T_j = T_j \circ T_i$.

[2] Even in this case we are forced to consider compact Hausdorff spaces.

2. Nilsystems

To show our result we need some tools. The first one is the known structure theorem.

Let G be a group. For $g, h \in G$, we write

$$[g, h] = ghg^{-1}h^{-1}$$

for the commutator of g and h and we write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$.

The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting

$$G_1 = G, \text{ and } G_{j+1} = [G_j, G].$$

Let $k \geq 1$ be an integer. We say that G is k-step nilpotent if G_{k+1} is the trivial subgroup.

2. Nilsystems

Let G be a k-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a k-step nilmanifold.

The group G acts on X. That is, for a fixed $\tau \in G$, define

$$T = T(\tau) : X \longrightarrow X, \ x\Gamma \mapsto (\tau x)\Gamma.$$

The Haar measure μ of X is the unique probability measure on X invariant under this action. Then (X, T, μ) is called a k-step nilsystem. When the measure is not needed for results, we omit and write that (X, T) is a k-step nilsystem.

A k-step pronilsystem is an inverse limit of k-step nilsystems. An ∞-step pronilsystem is an inverse limit of nilsystems.

2. The regionally proximal relation $\mathbf{RP}$

Let G be a group acting on X. (X, G) is **equicontinuous** if for each $\epsilon > 0$ there is $\delta > 0$ such that $\rho(x, y) < \delta$ implies that $\rho(gx, gy) < \epsilon$ for any $g \in G$.

$(x, y) \in \mathbf{RP}$ (**regionally proximal relation**) if for each neighbourhood $U \times V$ of (x, y) and $\epsilon > 0$ there are $(x', y') \in U \times V$ and $g \in G$ with $\rho(gx', gy') < \epsilon$. It is easy to see $\mathbf{RP}$ is a closed invariant relation.

It is known: if G is amenabel and (X, G) is minimal, then $\mathbf{RP}$ is an equivalence relation. $X/\mathbf{RP}$ is the maximal equicontinuous factor **(MEF).**

2. $\mathbf{RP}^{[d]}$

Now we explain how ∞-step pronilsystems are connected with minimal systems (we just state it for $\mathbb{Z}$-actions)

Definition (HKM, 2010)

Let (X,T) be a tds and $d \in \mathbb{N}$. The points $x, y \in X$ are said to be **regionally proximal of order d (along cubes)**, *denoted by $(x,y) \in \mathbf{RP}^{[d]}$ if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x,x') < \delta$, $\rho(y,y') < \delta$, and*

$$\rho(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta$$

for any $\epsilon = \{\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d \setminus \{0, \ldots, 0\}$, where $\mathbf{n} \cdot \epsilon = \epsilon_1 n_1 + \cdots + \epsilon_d n_d$.

Note that: $\mathbf{RP} = \mathbf{RP}^{[1]}$.

2. For $d = 1, 2$

- For $d = 1$ we need n_1 with

$$\rho(T^{n_1}x', T^{n_1}y') < \epsilon.$$

- For $d = 2$ we need n_1, n_2 with

$$\rho(T^{n_1}x', T^{n_1}y') < \epsilon,$$

$$\rho(T^{n_2}x', T^{n_2}y') < \epsilon,$$

$$\rho(T^{n_1+n_2}x', T^{n_1+n_2}y') < \epsilon.$$

2. For $d = 3$

♦ For $d = 3$ we need n_1, n_2, n_3 with

$$\rho(T^{n_1}x', T^{n_1}y') < \epsilon, \rho(T^{n_2}x', T^{n_2}y') < \epsilon,$$

$$\rho(T^{n_1+n_2}x', T^{n_1+n_2}y') < \epsilon, \rho(T^{n_3}x', T^{n_3}y') < \epsilon,$$

$$\rho(T^{n_1+n_3}x', T^{n_1+n_3}y') < \epsilon, \rho(T^{n_2+n_3}x', T^{n_2+n_3}y') < \epsilon,$$

and

$$\rho(T^{n_1+n_2+n_3}x', T^{n_1+n_2+n_3}y') < \epsilon.$$

2. Pro-nilfactors

It is known (by Host-Kra-Maass 2010, Shao-Ye 2012) that for a minimal system

1. $\mathbf{RP}^{[d]}$ is an equivalence relation, and has lifting property.
2. $X_\infty = X/\mathbf{RP}^{[\infty]}$ is the inverse limit of nilsystems. [3]

Let $\mathbf{RP}^{[\infty]} = \cap_{i=1}^{\infty} \mathbf{RP}^{[d]}$ and $X_\infty = X/\mathbf{RP}^{[\infty]}$, which is called the ∞-step pronilfactor of (X,T).

[3] It was proved in Host-Kra-Maass using ergodic method. It can also be proved using the theory of nilspaces, see the work by Candela, Gutman, Szegedy and the coauthors.

3. The factor map

The second tool we need is the so-called “saturation theorem”.

That is, we need a deep understanding of $\pi : X \longrightarrow X_\infty$.

Unlike the situation in ergodic theory, here we need a modification such that the resulting factor map is open.

3. The linear case

The saturation theorem was first proved for the linear case [4]

Theorem (Glasner-Huang-Shao-Weiss-Y., 2020)

Let (X,T) be minimal, and $\pi : X \to X_\infty$ be the factor map. Then there are minimal systems X^* and X^*_∞ (almost 1-1 extensions of X and X_∞), and a commuting diagram s.t. X^*_∞ is a d-step topological characteristic factor of X^* for all $d \geq 2$

$$\begin{array}{ccc} X & \xleftarrow{\ \sigma\ } & X^* \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi^*(open)} \\ X_\infty & \xleftarrow{\ \tau\ } & X^*_\infty \end{array}$$

[4] $\pi : X \to Y$ is almost 1-1 if $\{x \in X : |\pi^{-1}\pi(x)| = 1\}$ is a dense G_δ set.

We now explain the notions appeared in the above theorem.

Let $\pi : X \longrightarrow Y$. $A \subset X$ is **π-saturated** if $\pi^{-1}\pi(A) = A$.

For a t.d.s. (X, T) and $d \in \mathbb{N}$, let $\tau_d = T \times T^2 \times \cdots \times T^d$.

Given a factor map $\pi : (X, T) \to (Y, T)$ and $d \geq 2$, the t.d.s. (Y, T) is said to be a **d-step topological characteristic factor (along τ_d) of (X, T)**, if there exists a dense G_δ subset Ω of X such that for each $x \in \Omega$ the orbit closure

$$L_x = \overline{\mathcal{O}}(x^{(d)}, \tau_d) = \overline{\{(T^n x, T^{2n} x, \ldots, T^{dn} x) : n \in \mathbb{Z}\}} \subset X^d$$

is $\pi \times \cdots \times \pi$ (d-times) saturated.

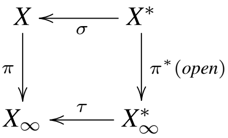
3. A saturation theorem of Qiu

Let $P = \{p_1, \ldots, p_d\}$ be distinct non-constant integral polynomials with $p_i(0) = 0$ for $1 \le i \le d$.

Theorem (Weak form of saturation for polynomials, Qiu, 2022)

Let (X, T) be minimal and $\pi : X \to X_\infty$ be the factor map. Then $\exists$ minimal X^* and X^*_∞ (almost 1-1 extensions of X and X_∞ resp.), and a commuting diagram s.t. for any open subsets V_i of X^* for $0 \le i \le d$ with $\bigcap_{i=0}^{d} \pi^*(V_i) \neq \emptyset$ and given P, $\exists\, n \in \mathbb{Z}$ and $x \in V_0$ with

$$T^{p_1(n)} x \in V_1, \ldots, T^{p_d(n)} x \in V_d.$$



Remark: It is used to solve the density problem.

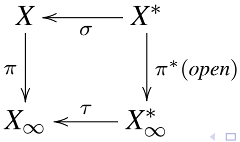
3. A saturation theorem for polynomials

Let $P = \{p_1, \dots, p_d\}$ be distinct non-constant integral polynomials with $p_i(0) = 0$ for $1 \le i \le d$.

Theorem (Saturation for poly., HSY, arXiv:2301.07873)

Let (X, T) be minimal and $\pi : X \to X_\infty$ be the factor map. Then $\exists$ minimal X^* and X^*_∞ (almost 1-1 extensions of X and X_∞ resp.), a commuting diagram below and a dense G_δ set Ω s.t. for each $x \in \Omega$, $d \in \mathbb{N}$ and open sets $U_1, \dots, U_d$ with $\pi^*(x) \in \cap_{i=1}^d \pi^*(U_i)$, there is $n \in \mathbb{Z}$ with

$$T^{p_1(n)}x \in U_1, T^{p_2(n)}x \in U_2, \dots, T^{p_d(n)}x \in U_d.$$



3. Remarks

We have the following remarks.

1. The saturation theorem for the linear case can be stated in the form of the one for general polynomials.
2. The proof of the saturation theorem is very long, I will not discuss it here. Roughly speaking, the theorem says that each minimal system is built by two parts: **"the structured part"** (∞-step pronil-system), and **"the random part"**.
3. The almost 1-1 modification is necessary by a result of Wu-Xu-Ye.

4. A system associated for polynomials

The final tool we need is the associated system related to a given t.d.s. and a finite collection of integral polynomials.

To show the dynamical version of our result, we need to pass it to pass it to another dynamical version.

4. A system associated for linear polynomials

Glasner (1994) introduced an associated system N_d for a t.d.s. Let (X,T) be a t.d.s. Set $\tau_d = T \times T^2 \times \cdots \times T^d$ and $T^{(d)} = T \times \cdots \times T$. Then

$$N_d(X,T) = \overline{\cup_{x \in X} \mathcal{O}(x^{(d)}, \langle T^{(d)}, \tau_d \rangle)},$$

where $x^{(d)} = (x, \ldots, x)$ (d-times), and $\langle T^{(d)}, \tau_d \rangle$ is the group generated by $T^{(d)}$ and τ_d. Note that if (X,T) is minimal, then for any $x \in X$,

$$N_d(X,T) = \overline{\mathcal{O}}(x^{(d)}, \langle T^{(d)}, \tau_d \rangle).$$

A deep result is the following

Theorem (Glasner)

If (X,T) is minimal, then so is $(N_d, \langle T^{(d)}, \tau_d \rangle)$.

Now assume that (X,T) is a t.d.s. and $p(n)=n^2$.

We define a system on $\subset X^{\mathbb{Z}}$

$$N_\infty(X,T) = \overline{\cup_{x\in X}\mathcal{O}((T^{n^2}x)_{n\in\mathbb{Z}}, \langle T^\infty,\sigma\rangle)}$$
$$= \overline{\cup_{x\in X}\{(\ldots, T^{m+(n-1)^2}x, \underset{\bullet}{T^{m+n^2}}x, T^{m+(n+1)^2}x,\ldots) : n,m\in\mathbb{Z}\}},$$

where

$$T^\infty = \cdots\times T\times T\times T\times\cdots,$$

and σ is the left shift.

4. A system associated for polynomials

Generally, for integral polynomials $\mathcal{A} = \{p_1, \ldots, p_d\}$ with $p_i(0) = 0$, a point of $(X^d)^{\mathbb{Z}}$ is denoted by

$$\mathbf{x} = (\mathbf{x}_n)_{n\in\mathbb{Z}} = \Big((x_n^{(1)}, x_n^{(2)}, \cdots, x_n^{(d)})\Big)_{n\in\mathbb{Z}}.$$

Let $\sigma : (X^d)^{\mathbb{Z}} \to (X^d)^{\mathbb{Z}}$ be the shift map, i.e., for all $(\mathbf{x}_n)_{n\in\mathbb{Z}} \in (X^d)^{\mathbb{Z}}$

$$(\sigma\mathbf{x})_n = \mathbf{x}_{n+1}, \ \forall n \in \mathbb{Z}.$$

Let (X, T) be a tds. For each $x \in X$, put

$$\omega_x^{\mathcal{A}} = \big((T^{p_1(n)}x, T^{p_2(n)}x, \ldots, T^{p_d(n)}x)\big)_{n\in\mathbb{Z}} \in (X^d)^{\mathbb{Z}}$$

Then

$$N_\infty(X, \mathcal{A}) = \overline{\{(T^\infty)^n\sigma^m(\omega_x^{\mathcal{A}}) : n, m \in \mathbb{Z}, x \in X\}} \subset (X^d)^{\mathbb{Z}}.$$

4. A system associated for polynomials

Remark

1. *It is clear that $N_\infty(X,\mathcal{A})$ is invariant under the action of T^∞ and σ, and $T^\infty \circ \sigma = \sigma \circ T^\infty$. Thus $(N_\infty(X,\mathcal{A}), \langle T^\infty, \sigma\rangle)$ is a $\mathbb{Z}^2$-t.d.s.*
2. *If (X,T) is transitive, then for each transitive point x of (X,T), $N_\infty(X,\mathcal{A}) = \overline{\mathcal{O}(\omega_x^{\mathcal{A}}, \langle T^\infty, \sigma\rangle)}$.*
3. *Sometimes we identify points in $(X^{d_1+d_2})^{\mathbb{Z}}$ as $(X^{d_1})^{\mathbb{Z}} \times (X^{d_2})^{\mathbb{Z}}$ as follows:*

$$\Big((x_n^{(1)},\dots,x_n^{(d_1+d_2)})\Big)_{n\in\mathbb{Z}} = \Big(((x_n^{(1)},\dots,x_n^{(d_1)}))_{n\in\mathbb{Z}}, ((x_n^{(d_1+1)},\dots,x_n^{(d_1+d_2)}))_{n\in\mathbb{Z}}\Big).$$

4. A system associated for polynomials

Examples: Let (X, T) be minimal.

(1) The case $\mathcal{A} = \{p\}$ with $p(n) = n$ we have

$$(N_\infty(X, \mathcal{A}), \sigma) \cong (X, T), \tag{4.1}$$

where $\cong$ means two systems are isomorphic.

(2) Let $\mathcal{A} = \{p_1, p_2, \ldots, p_d\}$, where $p_i(n) = a_i n, \ 1 \le i \le d$ and $a_1, a_2, \ldots, a_d$ are distinct non-zero integers. In this case we have

$$(N_\infty(X, \mathcal{A}), \sigma) \cong (N_{\mathcal{A}}(X, T), \tau_{\vec{a}}), \tag{4.2}$$

where $N_{\mathcal{A}}(X, T)$ is the orbit closure of $(x, \ldots, x)$ under $T \times T \times \cdots \times T$, and $\tau_{\vec{a}} = T^{a_1} \times T^{a_2} \times \cdots \times T^{a_d}$. It is known that $(N_{\mathcal{A}}(X, T), \langle T^\infty, \sigma\rangle)$ is minimal by Glasner.

4. A system associated for polynomials

(3) (X, T) is weakly mixing. If $\mathcal{A} = \{n^2\}$ we have

$$(N_\infty(X, \mathcal{A}), \sigma) \cong (X^{\mathbb{Z}}, \sigma).$$

We remark that $(N_\infty(X, \mathcal{A}), \langle T^\infty, \sigma \rangle)$ is not minimal in the above case.

If $\mathcal{A} = \{n, n^2\}$ we have

$$(N_\infty(X, \mathcal{A}), \sigma) \cong (X \times X^{\mathbb{Z}}, T \times \sigma), \tag{4.3}$$

4. A system associated for polynomials

(4) If (X, T) is ∞-step pronil, so is $(N_\infty(X, \mathcal{A}), \langle T^\infty, \sigma \rangle)$.

(5) If (X, T) is distal,

$$(N_\infty(X, \mathcal{A}), \langle T^\infty, \sigma \rangle)$$

may not be distal (unless it is ∞-step pronil).

4. A system associated for polynomials

Theorem

Let (X,T) be a minimal t.d.s. [a] *Then the following statements are equivalent:*

1. *For any family $\mathcal{A}=\{p_1,p_2,\cdots,p_d\}$ of integral polynomials with $p_i(0)=0$, $(N_\infty(X,\mathcal{A}),\langle T^\infty,\sigma\rangle)$ is an M-system.*
2. *The dynamical version of our result, i.e.*

   For any integral polynomials $p_1,\ldots,p_d$ with $p_i(0)=0$, $1\le i\le d$, we have that for each $x\in X$ and any neighbourhood U of x

$$\{(m,n)\in\mathbb{Z}^2 : T^{m+p_1(n)}x\in U,\ldots,T^{m+p_d(n)}x\in U\}\in\mathcal{F}_{ps}(\mathbb{Z}^2).$$

[a] (X,G) is an M-system if it is transitive and the set of minimal points is dense.

4. A system associated for polynomials

We remark that to use the saturation theorem we need to give a condition for the family $\mathcal{A}$.

Definition

Let $\mathcal{A} = \{p_1, \ldots, p_d\}$ be a family of integral polynomials. We say $\mathcal{A}$ satisfies condition (♠) if $p_i(0) = 0$ and

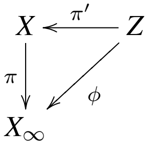
1. $p_i(n) = a_i n, 1 \leq i \leq s$, *where* $s \geq 0$, *and* $a_1, \ldots, a_s$ *are distinct non-zero integers;*
2. $\deg p_j \geq 2, s+1 \leq j \leq d$;
3. **for each** $i \neq j \in \{s+1, s+2, \ldots, d\}$, $p_j^{[k]} \neq p_i^{[t]}$ **for any** $k, t \in \mathbb{Z}$, **where** $p^{[j]}(n) = p(n+j) - p(j)$, $\forall n \in \mathbb{Z}$.

It is easy to see that for a given family $\mathcal{A}$ there is a maximal subfamily $\mathcal{A}'$ with the condition (♠).

5. The ideas of the proof

It is left to show for any family $\mathcal{A} = \{p_1, \cdots, p_d\}$ of integral polynomials with $p_i(0) = 0$ with the condition $(\spadesuit)$, $(N_\infty(X, \mathcal{A}), \langle T^\infty, \sigma \rangle)$ is an M-system.

To explain the main ideas of the proof, we start with a minimal distal system (X, T) and $\mathcal{A} = \{p\}$ with $deg(p) \geq 2$. Let $\mathcal{G} = \langle T^{\infty}, \sigma \rangle$.



We illustrate the idea by assuming that

(1) π is an equicontinuous extension.
(2) $\phi : Z \longrightarrow X_{\infty}$ is a group extension and
(3) the maximal ∞-step pro-nilfactor of Z is also X_{∞}.

5. The distal case

Then we have the following diagram.

$$
\begin{array}{ccc}
N_\infty(X,\mathcal{A}) & \xleftarrow{(\pi')^\infty} & N_\infty(Z,\mathcal{A}) \\
\downarrow{\scriptstyle \pi^\infty} & \swarrow{\scriptstyle \phi^\infty} & \\
N_\infty(X_\infty,\mathcal{A}) & &
\end{array}
$$

Note that $N_\infty(X_\infty, \mathcal{A})$ is ∞-step pro-nil and by the saturation theorem, for each $\mathbf{y} = (y_i)_{i\in\mathbb{Z}} \in N_\infty(X_\infty, \mathcal{A})$

$$(\phi^\infty)^{-1}(\mathbf{y}) = \prod_{i\in\mathbb{Z}} \phi^{-1} y_i.$$

5. The distal case

It suffices to show the minimal points of $\mathcal{G}$ is dense in $N_\infty(Z, \mathcal{A})$.

Now fix a point $\mathbf{y} = (y_i)_{i\in\mathbb{Z}} \in N_\infty(X_\infty, \mathcal{A})$ and $\mathbf{x} = (x_i)_{i\in\mathbb{Z}} \in (\phi^\infty)^{-1}(\mathbf{y})$. Since $\mathbf{y}$ is $\mathcal{G}$-minimal, there exists a $\mathcal{G}$-minimal point $\mathbf{x}' = (x_i')_{i\in\mathbb{Z}} \in (\phi^\infty)^{-1}(\mathbf{y})$.

Let $\phi_i \in Aut(Z)$ such that $x_i = \phi_i(x_i')$. For each $k \in \mathbb{N}$, let

$$\begin{aligned}\Phi_k &= (\phi_{-k} \times \phi_{-k+1} \times \cdots \times \phi_k)^\infty \\ &= \cdots \times (\phi_{-k} \times \phi_{-k+1} \times \cdots \times \phi_k) \times \\ &\qquad (\phi_{-k} \times \phi_{-k+1} \times \cdots \times \phi_k) \times \cdots .\end{aligned}$$

Then

$$\begin{aligned}\Phi_k(\mathbf{x}') = (\ldots, \phi_k(x'_{-k-1}), x_{-k}, \ldots, \underset{\bullet}{x_0}, \ldots, x_k, \\ \phi_{-k}(x'_{k+1}), \ldots, \phi_k(x'_{3k+1}), \phi_{-k}(x'_{3k+2}), \ldots) \\ \to \mathbf{x},\ k \to \infty.\end{aligned}$$

It is not difficulty to show that

1. $\Phi_k(\mathbf{x}') \in N_\infty(Z, \mathcal{A})$
2. $\Phi_k(\mathbf{x}')$ is $\mathcal{G}$-minimal.

This shows that the $\mathcal{G}$-minimal points are dense in $N_\infty(Z, \mathcal{A})$.

5. The ideas of the proof

The idea of the proof of the general case is similar to the above, but is much involved.

Let $\mathrm{Aut}(X, T)$ be the group of automorphisms of the t.d.s. (X, T), that is, the group of all self-homeomorphisms ψ of X such that $\psi \circ T = T \circ \psi$.

For an extension $\pi : (X, T) \to (Y, T)$, let

$$\mathrm{Aut}_\pi(X, T) = \{S \in \mathrm{Aut}(X, T) : \pi \circ S = \pi\},$$

i.e., the collection of elements of $\mathrm{Aut}(X, T)$ mapping every fiber of π into itself.

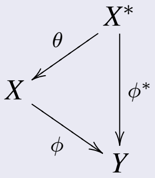
5. The ideas of the proof

Definition

Let $\pi : (X,T) \to (Y,T)$ be an extension of minimal t.d.s. One says π is regular if for any minimal point (x_1,x_2) in R_π (i.e. $\pi(x_1)=\pi(x_2)$) there exists $\chi \in \mathrm{Aut}_\pi(X,T)$ s.t. $\chi(x_1)=x_2$.

Theorem (Vries's book)

Let (X,T) and (Y,T) be minimal t.d.s. and let ϕ be the factor map. Then there is an extension $\theta : (X^*,T) \to (X,T)$ such that $\phi^* = \phi\circ\theta : (X^*,T) \to (Y,T)$ is regular.



5. The ideas of the proof

Let $\pi : (X,T) \to (Y,T)$ be a factor map of minimal t.d.s., and $x_0 \in X$, $y_0 = \pi(x_0)$. Let $u \in J$ such that $ux_0 = x_0$. We say that π is a RIC (relatively incontractible) extension if for every $y = py_0 \in Y$, $p \in \mathbf{M}$,

$$\pi^{-1}(y) = p \circ \left(u\pi^{-1}(y_0)\right). \tag{5.1}$$

where

$$p \circ A = \{x \in X : \forall\ \lambda \in \Lambda \text{ there is } d_\lambda \in A \text{ with } x = \lim_{\lambda} m_\lambda d_\lambda\}$$

for any net $\{m_\lambda\}_{\lambda \in \Lambda} \subseteq \mathbb{Z}$ converging to p.

Note that every distal extension is RIC, and every RIC extension is open.

5. The ideas of the proof

Every factor map between minimal systems can be lifted to a RIC extension by proximal extensions.

Theorem (EGS)

Given a factor map $\pi : (X, T) \to (Y, T)$ of minimal systems, there exists a commutative diagram of factor maps (called RIC-diagram)

$$\begin{array}{ccc} X & \xleftarrow{\ \theta'\ } & X^* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & \xleftarrow{\ \theta\ } & Y' \end{array}$$

such that:

(a) θ' *and* θ *are proximal extensions;*

(b) π' *is a RIC extension;*

5. The ideas of the proof

The proof is done in four steps:

1. firstly we construct the extension using the regularizer and the universal minimal system (**Step 1**),
2. secondly we transfer the question into some extension of $(N_\infty(X^*, \mathcal{A}), \langle T^\infty, \sigma\rangle)$ (**Step 2**);
3. then we show the set of minimal points is dense in a certain region (**Step 3**);
4. and finally we use the RIC property to spread minimal points to the whole space (**Step 4**).[5]

---

[5] The weakly mixing extension makes trouble here.

5. Some open questions

We just proved if $F \in \mathcal{F}_{ps}(\mathbb{Z})$ then

$$\{(m,n) \in \mathbb{Z}^2 : m + p_1(n), \ldots, m + p_k(n) \in F\} \in \mathcal{F}_{ps}(\mathbb{Z}^2).$$

The first question is that

Question

What is the relationship between two piece-wise syndeticity?

5. Some open questions

Question

Assume that $d \in \mathbb{N}$, $S \in \mathcal{F}_{ps}(\mathbb{Z}^2)$, *and* $p_{i,j}$ *is an integral polynomial with* $p_{i,j}(0) = 0$ *for each* $1 \le i \le d, 1 \le j \le 2$. *Consider the set*

$$\{(m_1, m_2, n) \in \mathbb{Z}^3 : (m_1 + p_{1,1}(n), m_2 + p_{1,2}(n)) \in S, \ldots, (m_1 + p_{d,1}(n), m_2 + p_{d,2}(n)) \in S\}.$$

Is it true that the above set is piecewise syndetic in $\mathbb{Z}^3$ *?*

Remark: Our method does not work for this situation. And we guess that there is a combinatoric proof of our result which can be used to solve the question, even for the nilpotent actions.

5. Some open questions

A general question is

Question

How to show the saturation theorem for other subsets of $\mathbb{Z}$? For example, the set of primes.

谢谢！