Chapter 1 :

Chain recurrence classes ,

Conley theory,

the fundamental theorem of dynamical systems.

References :
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1) Introduction: the spectral decomposition theorem.

In his paper "Differentiable dynamical systems" Smale defines a class of diffeomorphisms *f*, open for any C^r topology, $r\geq 1$, called *Axiom A + the no cycle condition.* This class is an important class because it is precisely the class whose recurrence set is structurally stable, we will come back to this property later.

For describing the dynamics he splits the reccurrence set in finitely many pieces K_i , $i=1, \dots, k$ which are :

(Smale's spectral decomposition theorem)

- compact, disjoint
- *transitive* (there is a dense orbit in each class) *topologically ergodic* (there is a dense positive orbit in each class). Most of the authors now use *transitive* for *topologically ergodic*, and I will do so.
- There is a filtration $\emptyset=M_0\subset \text{Int } M_1 \subset M_1\subset\dots\subset M_k=M$ so that $f(M_i)\subset \text{Int}(M_i)$ and
- K_i is the maximal invariant set in $M_i\setminus M_{i-1}$, that is, every orbit contained in $M_i\setminus M_{i-1}$ is contained in K_i. In formula this gives :
- Every point x not contained in the union of the K_i is wandering : it admits a neighborhood V_x which is disjoint from all its iterates $f^n(V_x)$.

In particular the K_i are maximal (for the inclusion) transitive (i.e. topologically ergodic) sets.

Exercize 1

Prove that a homeomorphism h of a compact metric space is *transitive* (meaning *topologically ergodic*) if and only if given any two non-empty open subset U,V there is n>0 with $h^n(U)\cap V\neq\emptyset$

Exercize 2

let h *be a homeomorphisms of a compact metric space. Shows that every recurrent orbit is contained in a maximal transitive compact set*.

Conley shows that this splitting of the dynamics can be generalized in a very general setting, not needing any hyperbolic structure, but changing the notion of recurrence one considers.

Exercize 3:
give examples of homeomorphism of a compact set having two maximal transitive subsets with non-empty intersection. Can you realize it as a smooth diffeomorphism of a closed manifold?

Exercize3 shows that transitivity needs to be changed by another notion, if we want a generalization beyond uniform hyperbolicity.

2) Notion of pseudo orbit, chain recurrence, chain recurrence classes.

Let h be a homeomorphism of a compact metric space X.
Given $\varepsilon>0$, an ε-*pseudo orbit* is a sequence $x_0, .. x_n$, $n>0$, so that $d(h(x_i),x_{i+1}) < \varepsilon$.
A point x is said *chain recurrent* if for any $\varepsilon>0$ there is a ε-pseudo orbit $x=x_0,..x_n=x$, $n>0$.
We denote by $\mathcal{R}(h)$ the set of chain recurrent points of h

exercize 4 prove that $\mathcal{R}(h)$ is a compact invariant subset of h.

exercize 5 Consider the map $h:t\rightarrow t+(1/4\pi)(1-\cos 2\pi t)$ of the circle $\mathbb{R}/\mathbb{Z}$. Show that it is a diffeomorphism of the circle. Show that it has a unique maximal transitive set K(h) and determine it. What is $\mathcal{R}(h)$?

Given $\varepsilon>0$ one defines a relation on $\mathcal{R}(h)$ by $x\dashv_\varepsilon y$ if there is a pseudo-orbit starting at x and ending at y.

One denotes $x\dashv y$ if $x\dashv_\varepsilon y$ for every $\varepsilon>0$.
We denote $x\dashv\vdash y$ if $x\dashv y$ and $y\dashv x$

Theorem 1 the relation $\dashv\vdash$ induces an equivalence relation on $\mathcal{R}(h)$ whose equivalence classes are compact invariant subsets.

Exercize 6: prove Theorem 1, it is truly easy.

The equivalence classes for $\dashv\vdash$ are called the chain recurrence classes.

3) The fundamental theorem of Dynamical Systems.

Theorem Let $f: X\rightarrow X$ be a homeomorphism of a compact metric space. Then there is a continuous function $\varphi: X\rightarrow\mathbb{R}$ so that

- for every point x, $\varphi(f(x))\leq\varphi(x)$ in other words, φ decreasing along the orbits. We say that $\varphi: X\rightarrow\mathbb{R}$ is a *Lyapunov function*
- $\varphi(f(x))=\varphi(x)$ if and only if x is chain recurrent (x in $\mathcal{R}(h)$)
- for x,y in $\mathcal{R}(h)$, $\varphi(x)=\varphi(y)$ if and only if x and y are in the same chain recurrence class, that is $x\dashv\vdash y$
- the (compact) subset $\varphi(\mathcal{R}(h))$ of $\mathbb{R}$ has empty interior, in other words it is totally disconnected: between the values of 2 distinct chain recurrence classes there is a level with no recurrence classes.

Furthermore, if X is a closed manifold, then φ can be chosen to be smooth.

The interest of the last item is that 2 any classes are separated by a regular level, hence bt a filtration. The filtration in Smale spectral decomposition theorem is given by cutting the manifold along regular levels between the level of the different classes.

4) Pairs attractor repeller

A *trapping region* is a compact set U so that $f(U) \subset Int(U)$.

The maximal invariant set $A_U = \cap^{\infty}_{-\infty} f^n(U)$ is called an *attractor*. A_U is a compact invariant set so that for every $x \in U$ the ω-limit set ω(x) is contained in A_U.

Exercize 7 Every compact set V contained in U and containing f(U) in Int(V) is a trapping region for the attractor A_U.

A *repeller* is an attractor for f^{-1}
exercize 8 If U is a trapping region for f, X\ Int U is a trapping region for f^{-1}.

A pair (attractor, repeller) is a pair (A,R) so that there is a trapping region U for A for which X\ Int U is a trapping region for (f^{-1}, R).

The main step for the fundamental Theorem is

Theorem 2 Let $f:X \to X$ be a homeomorphism of a compact metric space. Then

- a point x is chain recurrent if and only if, for every pair (A,R) of attractor repeller one has

$$x \in A \cup R$$

In other words one has

$$\mathcal{RA}(h) = \mathcal{A}(h) = \bigcap_{\{pairs\ (A,R)\}} A \cup R$$

- if $x,y \in \mathcal{R}(h)$ then $x \dashv\vdash y$ if and only if for any pair (A,R) of attractor repeller one has

$$x \in A \text{ if and only if } y \in A$$

one denote $x \approx y$

Lemma *Assume that* (A,R) *is a pair attractor repeller and* $x \notin A \cup R$. *Then* $x \notin \mathcal{R}(h)$.
proof
Assume $x \notin A \cup R$ where A is the maximal invariant in an attracting region U. Then there is n so that $x \in h^n(U)$ but $x \notin h^{n+1}(U)$. Consider d less than ½ inf $(d(x, h^{n+1}(U))\ \ d(X \setminus h^n(U), h^{n+1}(U))$.
Consider a d-pseudo x_0 x_1 orbit starting at x.
Then $h(x) \in h^{n+1}(U)$ so $x\_1 \in h^n(U)$ and at distance at least d from x. Thus $h(x\_1) \in h^{n+1}(U)$ so $x\_2 \in h^n(U)$, and at distance at least d from x. Ans by induction $h(x\_i) \in h^{n+1}(U)$ so $x\_i+1 \in h^n(U)$, and at distance at least d from x.
□
We just proved $\mathcal{R}(h) \subset \mathcal{A}(h)$.

The main idea for the proof of <== in theorem Theorem 2 is next lemma:

Lemma *Given any point* x *let* $W^+_\varepsilon(x)=\{y, x\dashv_\varepsilon y\}$. *Then the closure of* $W^+_\varepsilon(x)$ *is an attracting region.*

Proof: we check that the ε/2-neighborhood of the closure of $h(W^+_\varepsilon(x))$ is contained in $W^+_\varepsilon(x)$. Consider a point z in the closure of $W^u_\varepsilon(x)$. There are therefore points y_n tending to z and which are end points of ε-pseudo orbits starting at x. Then the ε-neighborhood of $h(y_n)$ is contained in $W^+_\varepsilon(x)$. For n large, the ε-neighborhood of $h(y_n)$ contains f(z). Thus the image of the closure of $h(W^+_\varepsilon(x))$ is contained in $W^+_\varepsilon(x)$.

□

In the same way one defines $W^-_\varepsilon(x)=\{y, y\dashv_\varepsilon x\}$, and *the closure of* $W^-_\varepsilon(x)$ *is an repelling region.*

Lemma x belongs to $\mathcal{R}(h)$ if and only if it belongs to $W^+_\varepsilon(x)$ for every ε>0, (or equivalently if and only if it belongs to $W^-_\varepsilon(x)$ for every ε>0).

Let $x\notin\mathcal{R}(h)$, and let A be the attractor associated to $W^+_\varepsilon(x)$. In particular $x\notin A$. Let R be the corresponding repeller. Thus R is the maximal invariant set in the complement of $W^+_\varepsilon(x)$ which is also the maximal invariant set in $h^{-1}(W^+_\varepsilon(x)) \subset W^+_\varepsilon(x)$. However $x \in h^{-1}(W^+_\varepsilon(x))$. Thus $x\notin R$. We just proved $x\notin A\cup R$ and therefore $x\notin\mathcal{A}$.

This proves $\mathcal{A}(h) \subset \mathcal{R}(h)$

Thus we proved

$$\mathcal{A}(h)=\mathcal{R}(h)$$

□

Consider now x,y in $\mathcal{R}(h)$. If x and y are not equivalent then there is ε>0 so that either one cannot get from x to y or from y to x by ε-pseudo orbits. Let assume the first. Thus $y \notin W^+_\varepsilon(x)$, in particular y is not in the attractor associated to $W^+_\varepsilon(x)$ when x belongs to this attractor.

Thus $x\approx y \Rightarrow x \dashv\vdash y$.

Conversely assume $x \not\approx y$ so that there is (A,R) a pair of attractor associating to a trapping region U with x in A and y in R (or conversely). Consider d smaller that d(U, h(U)). Now any ½ d-pseudo orbit starting in A is disjoint from $h^{-1}(U)$, and so cannot go to R: one just proved

$$x \dashv\vdash y \Rightarrow x\approx y.$$

This ends the proof of the Theorem 2

□

<u>5) Building Lyapunov functions</u>

Theorem Let $f:X\to X$ be a homeomorphism of a compact metric space and assume that (A,R) is a pair of attractor repeller. Then there is Lyapunov function $\varphi: X\to \mathbb{R}$ so that

- $\varphi(A)=0$ $\varphi(R)=1$
- for any $x \notin A\cup R$ one has $\varphi(f(x))<\varphi(x)$

Furthermore, if X is a closed manifold, then one can choose the map φ to be smooth.

proof:

Let U be a trapping region for (A,R). Then there is a map $\varphi_0: X\to [0,1]$ (smooth if X manifold) so that :

- $\varphi_0(X\backslash U)=1$
- $\varphi_0(f(U))=0$
- and φ_0: $Int(U)\backslash f(U) \to (0,1)$.

Note that φ_0 is decreasing along the orbits : it is a Lyapunov function.

One denotes φ_n: $X \to [0,1]$ defined by $\varphi_n(x)=\varphi_0(f^{-n}(x))$. It is a similar function just substituting U by $f^n(U)$ (if we want smooth,one just choose any Lyapunov function as φ_0 but replacing U by $f_n(U)$.

One choose a sequence $a_n>0$ to that $\Sigma a_n=1$, and consider

$$\varphi = \Sigma\, \varphi_n$$

- It is a continuous function, as the sum converges uniformly.
- $\varphi(A)=0$ $\varphi(R)=1$ because $\varphi_n(A)=0$ $\varphi_n(R)=1$.
- It decreases along the orbits because every φ_n decreases along the orbits.

Assume x is not in $A\cup R$. Thus there is n(x) so that:

- $f^i(x) \in X\backslash Int\, U$, if $i\leq n(x)$
- $f^{n(x)+1}(x) \in U$ and thus:
- $f^{n(x)+2}(x) \in f(U)$, and $f^j(x) \in f(U)$ for $j>n(x)+1$.

Thus

- $\varphi_i(x)=1$ if $i\leq n(x)$, $\varphi_j(x)=0$ if $j > n(x)+1$ and $\varphi_{n+1}(x) \in [0,1]$.
- $\varphi_i(f(x))=1$ if $i\leq n(x)-1$, $\varphi_j(x)=0$ if $j > n(x)$ and $\varphi_n(f(x)) \in [0,1]$.
- Moreover $\varphi_n(f(x)) =0$ (resp.1) if and only if $\varphi_{n+1}(x) =0$ (resp.1).

Thus

$$\varphi(x)= \Sigma_{-\infty}^{n(x)} a_i + a_{n(x)+1}\varphi_{n+1}(x) \quad \text{and} \quad \varphi(f(x))= \Sigma_{-\infty}^{n(x)-1} a_i + a_{n(x)}\varphi_n(f(x)).$$

So

$$\varphi(x)-\varphi(f(x))= a_{n(x)}\,(1-\varphi_n(f(x))) + a_{n(x)+1}\varphi_{n+1}(x) \quad >0$$

For the smoothness we just need to take care that the sequence a_n decrease fast enough so that the sum converges for the C^∞-topology.

□

Proposition : The set of pairs (A,R) attractor repeller is at most countable.

proof: let O_n be a countable basis of the topology of X, that is, every open set is a union of a subfamily of the O_n.

Lemma: A admits a trapping region which is the union of a finite subfamily of O_n

Let finish the proof using the lemma: there are at most countably many such finite union. So the set of such trapping region are countably many, and so do the corresponding attractor repeller associated pairs.

□

proof of the lemma : Int(U) is union of a subfamily of the O_n. This defines an open family covering f(U). One extract a finite cover using the compactness of f(U). The union of the open subsets in this finite cover is the announced trapping region.

□

6) proof of the Fundamental Theorem of Dynamical Systems:

One choses an indexation (A_n,R_n) of the countable set of the pairs attractor repeller, and we choose for each pair a Lyapunov function ψ_n given by the Theorem of the previous section.

One chooses a sequence $b_n>0$ with $\Sigma b_n=1$ (and so that $\Sigma \psi_n$ converges in the C^∞-topology), and so that $\Sigma_{i>n}b_i < 1/3\ b_n$. This implies that the maps $\theta:\{0,1\}^{\mathbb{N}}\to[0,1]$ mapping (δ_i) on $\Sigma\ \delta_n\ b_n$ has a totally discontinuous image.

Then $\psi=\Sigma\,\psi_n$ is a Lyapunov function. If x is not chain recurrent then there is n so that x does not belong to $A_n \cup R_n$ and thus $\psi_n(f(x))< \psi_n(x)$ and therefore $\psi(f(x))< \psi(x)$

If x,y are chain recurrent and in the same class then the $\psi_n(x)= \psi_n(y)$ for every n so that $\psi(x)= \psi(y)$.
Otherwise let x,y both chain recurrent but not in the same class. Notice that every $\psi_n(x)$, $\psi_n(y)$ has value 0 or 1. One consider the smallest n so that x,y are not both in the attractor or both in the repeller, for instance x in the repeller and y in the attractor.
Then $\psi(x)-\psi(y) \geq b_n- \Sigma_{i>n}b_i > 1/3\ b_n$ this shows that the images are distinct. This proved the 3 first items.

Item 4 comes from the fact that the images of any chain recurrence class belongs to the image of the maps $\theta:\{0,1\}^{\mathbb{N}}\to[0,1]$ which is totally discontinuous.

□

7)**The chain recurrent set and Smale spectral decomposition theorem**

Theorem f:M→ M diffeomorphism of a closed manifold. Then
f is Axiom A+ no-cycle condition <=> $\mathcal{R}(f)$ is hyperbolic.

Not at all elementary: Anosov closing lemma, Hayashi connecting lemma version B-,Crovisier.

Just notice that if f Axiom A + cycle, then the “cycle” is contained in $\mathcal{R}(f)$ and in fact in one class, and the hyperbolicity implies that there is in fact a unique homoclinic class.

8) Uncountably many classes for Analytic diffeomorphism

Theorem *Consider any family of diffeomorphisms* f_t , t *in* R, *of diffeomorphisms of the sphere* $\boldsymbol{S}^2$, *unfolding a homoclinic tangency at a point with jacobian <1.*

Then there is an open set in the space of parameters where generic parameters corresponds to diffeomorphisms with uncountably many chain recurrence classes which are adding machines.

Proof: each time you unfold a homoclinic tangency, you create a trapping region with a saddle point with jacobian <1 and unfolding a homoclinic tangency.
One creates therefore a tree of nested periodic trapping regions and each intersection of a branch is a adding machine.

□

The same proof provides a residual subset in any “Newhouse regions” (C^2 open set of surface diffeomorphisms displaying a robust tangency in a hyperbolic saddle like set) having uncountably chain recurrence classes which are adding machines.

9) Dynamics of generic homeomorphisms (Akin Hurley Kennedy)

The same argument providing uncountably many chain recurrence classes which are adding machines for locally generic parameters of an unfolding of a homoclinic tangency leads to the following:

Theorem Let M be a smooth closed manifold. There is a residual subset $\mathcal{G}$ of Homeo(M) so that f in $\mathcal{G}$ has dense subset in $\mathcal{R}(f)$ consisting in chain recurrence classes which are adding machines.

Proof: $\mathcal{R}(f)$ varies upper semi-continuously with f. Then generically it varies continuously. Small C^0 perturbations creates periodic points with non 0 Poincaré Hopf index close to any point in $\mathcal{R}(f)$ => generically $\mathcal{R}(f)$ is the cosure of the periodic point. Each periodic point leads to a periodic disc, by small perturbation. Periodic discs leads to trapping region of large period and small diameter, with an arbitrary dynamics of the return map… thus generically trees of nested sequence of trapping or repelling regions, and the intersections of any branch is an adding machine.

□

More generally, Akin Hurley Kennedy consider a compact, piecewise linear manifold M of dimension at least 2 with no boundary, and they show that

Theorem given a generic homeomorphism f of M:

1. $\mathcal{R}(f)$, the chain recurrent set for f , is a Cantor set.
2. The complement of the periodic points in $\mathcal{R}(f)$ is residual in $\mathcal{R}(f)$
3. $\mathcal{R}(f)=\Omega(f)$ the nonwandering set.
4. There are uncountably many chain recurrent classes which are adding machines and whose union is dense in $\mathcal{R}(f)$.
5. There are chain recurrence classes semi conjugated to a subshift of finite type
6. There is a residual subset of M consisting of points whose ω-limit and α-limit sets are each a chain recurrence class which is an adding machine
7. There is a residual subset of $\mathcal{R}(f)$ consisting of points that are Lyapunov stable

10) Open questions

Consider diffeomorphism f on a manifold. Let ψ be a Lyapunov function of f

Question *What are the critical points of the smooth Lyapunov functions*?

Remark: each hyperbolic periodic orbit is critical point.

Assume f is Morse-Smale. Then
– there are Morse-Lyapunov functions
– Pixton built a Morse-Smale diffeomorphism f on S^3 so that any Morse Lyapunov function have critical points distinct from periodic points of f.

Work in progress with some preprints by (Medvedev T. V., Nozdrinova E. V., Pochinka O. V.)

Solution of the exercises

1. If h admits a dense positive orbits it passes in U then in V so that the positive iterates of U meet V.
   Conversely, if given every two open subsets U,V there is a positive iterate of U meeting V, then the set O(V) of points whose positive orbit meets V is open and dense in K. Now we choose a countable basis {Vn} of the topology of K and we consider the countable intersection of the O(Vn). It is a dense G-delta of K, whose positive orbits are dense in K. A similar arguments provides a dense G-delta of K, whose negative orbits are dense in K so that the intersection of these two dense G-delta is a dense G-delta consisting of points whose both positive and negative orbits are dense.

2. Consider the set $\mathbb{K}$ of the compact transitive (topologically ergodic) sets ordered by the inclusion. Consider a totally ordered family of compact transitive sets Ki. Consider K the closure of the union of the Ki. Let U,V be two non-empty open subsets of K. Then the intersections Ui,Vi with Ki are non-empty open subsets for i large enough. Thus there is a positive iterate hn(Ui) intersecting Vi. Thus hn(U) intersects V. We just proved that K is topologically ergodic.
   So $(\mathbb{K}, \subset)$ is inductive and Zorn lemma implies the existence of a maximal element.

3. Consider the shift with 3 symbols, 0,1,2. Inside it consider the union of the subsets S_1, S_2 which are the shifts with 2 symbols, 0,1 and 0,2 respectively. The maximal transitive sets are S_1 and S_2 whose intersection is the constant sequence ...00000…

4. Use the uniform continuity of h for showing that the image of a pseudo orbit is a pseudo orbits, just changing the constant. This proves the invariance. For the compactness, a closed pseudo orbit at a point very close to x is a closed pseudo orbit at x, just chaging slightly the constant. As it holds for every constant, one can conclude.

5. The derivative is larger that ½ so that it is a local diffeomorphisms and therefore it is a smooth covering map of the circle. Changing smoothly the constant to 0, one shows that it is isotopic to the identity, so that its topological degree is 1 proving that it is a diffeomorphisms.
6. The symmetry is by definition. The reflexivity is the definition of $\mathcal{R}$(h) . The transitivity is by concatenation of the pseudo orbits. The invariance and the compactness of the classes is as the invariance and the compactness of $\mathcal{R}$(h) . Notice that f(x) $\dashv$ x if and only if x belongs to $\mathcal{R}$(h) .

Chapitre 2

Hyperbolicity and beyond.

Reference book: Palis, de Melo

1) Hyperbolicity I guess that everybody here has some notion of hyperbolicity. It started with linear algebra:

exercise 1 Consider matrices A, B $\in GL(n,\mathbb{R})$ acting as a diffeomorphisms of $\mathbb{R}^n$. Assume that no eigenvalue neither of A nor of B is of modulus =1. Give a necessary and sufficient condition for A and B being conjugated by a homeomorphism of $\mathbb{R}^n$. Under what condition the conjugacy homeomorphisms can be a diffeomorphism?

Exercize 2 Assume that A has a eigenvalue of modulus 1. Show that A is the limit of matrices which are not conjugated to A by homeomorphisms.

One says that a matrix A is *hyperbolic* if A has no eigenvalue of modulus 1.

A periodic orbit of a diffeomorphisms is hyperbolic if its derivative at the period is hyperbolic.

Theorems: local conjugacy to the linear part: Hartman Grobman, invariant manifolds (stable unstable, strong stable/unstable, local structural stablility etc… Kupka-Smale

The existence of invariant manifolds was known from Hadamard in 1898, and he makes reference to Poincaré 1891 for these invariant manifolds. Another reference could be Darboux 1878.

Beyond the existence of the invariant manifolds, Poincaré in 1887 (dans les nouvelles méthodes noticed that a transverse intersection of the stable and unstable manifold of the same hyperbolic periodic orbits leads to complicated behavior.

" When one tries to depict the figure formed by these two curves and their infinity of intersections, each of which corresponds to a doubly asymptotic solution, these intersections form a kind of net, web, or infinitely tight mesh; neither o f the two curves can ever intersect itself, but must fold back on itself in a very complex way in order to intersect all the links of the mesh infinitely often. One is struck by the complexity of this figure that I am not even attempting to draw. Nothing can give us a better idea of the complexity of the three-body problem and in general all the problems of dynamics where there is no single-valued integral and Bohlin's series diverge. “

The complexity of homoclinic intersection has been studied in particular by Birkhoff 1935 who proved that any transverse homoclinic intersection point is the limit of periodic points whose period tends to infinity.

In my sense the true reason of this complexity is explained by Smale, proving that every diffeomorphism admitting a transverse homoclinic intersection associated to a hyperbolic saddle periodic orbit has an iterate with an invariant hyperbolic set conjugated to his horseshoe. This not only shows the complexity of the dynamics but provides a rigid structure.

Definition Let f be a diffeomorphism of a manifold M, endowed with a riemannian metric $\|\ \|$.
A compact set K is said hyperbolic if

- K is invariant under f $f(K)=K$.
- there is n>0 and, at each $x \in K$ there are $E^s(x)$, $E^u(x)$ so that
- $T_xM=E^s(x) \oplus E^u(x)$
- $E^s(f(x))= T_xf(E^s(x))$, $E^u(f(x))= T_xf(E^u(x))$ (i.e. the bundles are f-invariant)
- for any vectors $u \in E^s(x)$, $v \in E^u(x)$ one has $\| T_xf^n(u)\| \leq ½ \|u\|$ and $\| T_xf^{-n}(v)\| \leq ½ \|v\|$.

exercise show that the vector subspaces $E^s(x)$, $E^u(x)$ are uniquely defined, and depend continuously on x.

Theorem (structural stability of hyperbolic sets) Assume K is a hyperbolic set. Then there is a neighborhood $\mathcal{U}(f)$ for the C^1-topology and continuous map ψ: $\mathcal{U}(f)\mathrm{x}K \to M$ $(g,x) \to \psi_g(x)$ so that ψ_f is the identity map of K and so that
– ψ_g is a homeomorphism of K on its image $\psi_g(K)$
– $\psi_g(K)$ is a g-invariant compact set
– the restriction of g to $\psi_g(K)$ is $\psi_g f \psi_g^{-1}$

Theorem (stable and unstable manifolds) Assume that K is a compact hyperbolic f-invariant set. Then there is $\varepsilon>0$ so that given any $x \in K$, the set of points y whose positive iterates remain at a distance $< \varepsilon$ from those of x is a compact disc $W^s_\varepsilon(x)$ tangent to $E^s(x)$ as smooth as f and varying continuously with x in the C^r topology.

Axiom A + "strong transversality" <=> C^1-structural stablility (Robin Robinson 1975; Mané 1988)

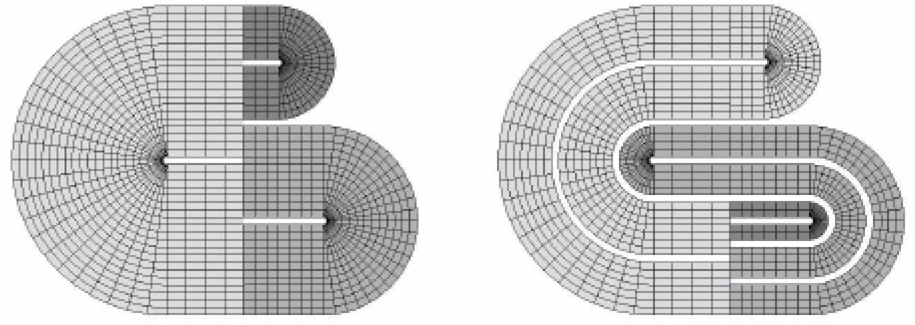
Axiom A+no cycle <=> C^1-Ω-stablity <=> $\mathcal{R}(f)$ is hyperbolic.

I cannot give a complete course on hyperbolic dynamics. So let me present you an open set of non-hyperbolic diffeomorphisms.

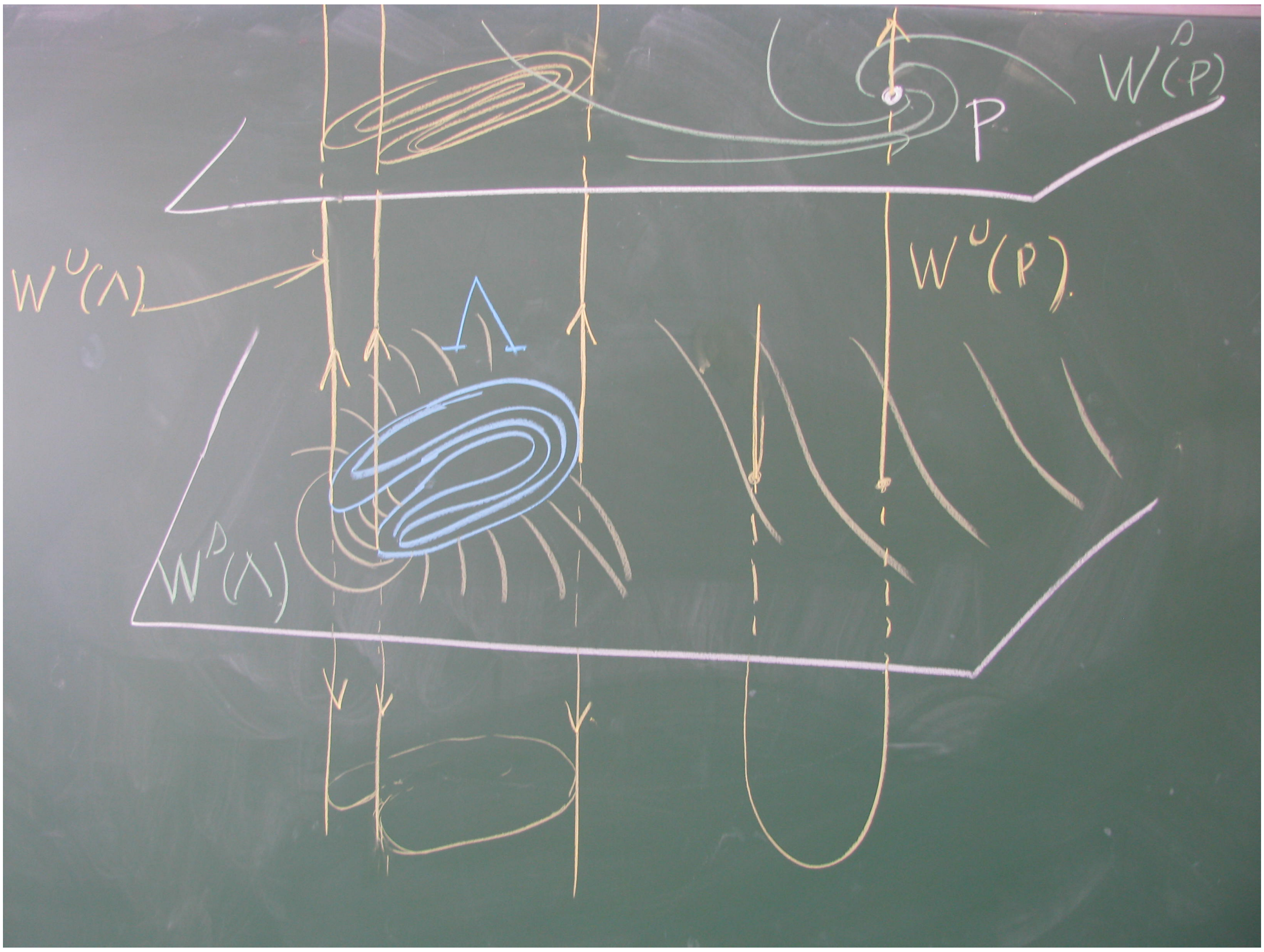
Still open Question "Smale conjecture": are the Axiom A+no-cycle diffeomorphisms dense in $\mathrm{Diff}^1(S)$ for S compact surface?

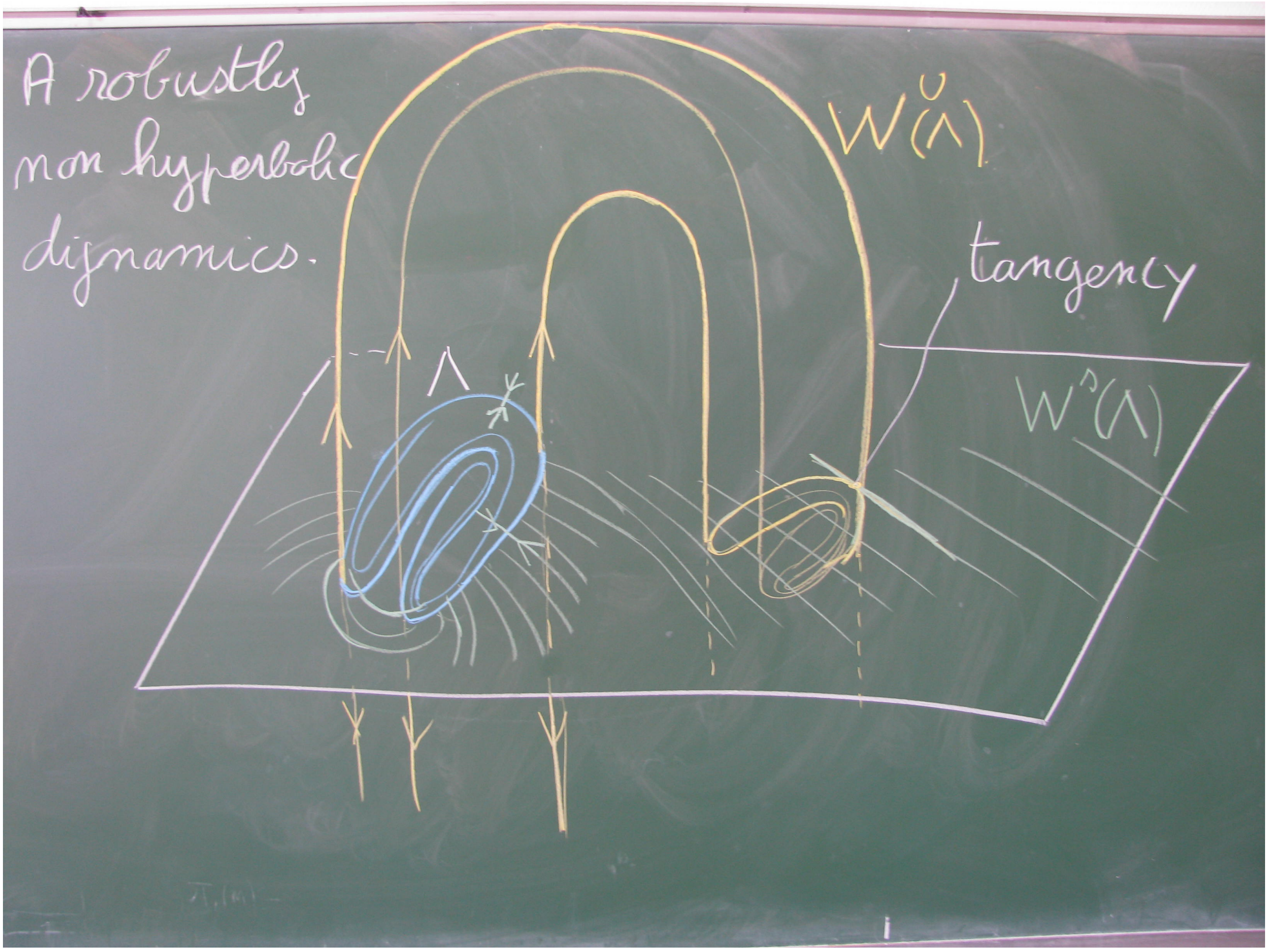
2) Robustly non-hyperbolic dynamics

We start with a hyperbolic attractor in the plane, Plykin attractor.



Let me show you now an open set of non-hyperbolic diffeomorphisms:





A C1-open set of diffeomorphisms with a chain recurrence class class with no dominated splitting.

In the picture above it is enough to add a point in the Plykin attractor, whose unstable eigenvalue is non-real : this breaks the possibility of any dominated splitting on the class of the splitting.
We will that this kind of dynamical systems are the most complicated which exists.

Definition: f $M \to M$ is said to have *(positive) universal dynamics* if there are discs $\varphi_i: D^d \to D\_i \subset M$, with the following properties:

- for every i there is $n_i>0$ with $f^{n_i}(D_i) \subset \mathrm{Int}(D_i)$, with $f^j(D_i))$ disjoint from D_i for $j<n_i$.
- The positive orbits of the D_i are pairwise disjoint
- the set $\{\varphi_i^{-1} f^{n_i} \varphi_i\}$ is dense in Diff $(D^d, \mathrm{Int}\, D^d)$

One defines in the same way *negative universal dynamics*, and *universal dynamic* is positive and negative universal dynamics.

Theorem Let U be a C1 open set of diffeomorphism on M, dim M= 3, so that

- $f \in U \to p_f, q_f$ hyperbolic periodic orbit varying continuously with f
- and $\dim W^s(p_f) = \dim W^u(q_f) = 2$
- $p_f \dashv\vdash q_f$ (they share the same chain recurrence class)
- the class of p_f, and q_f contains periodic points with complex stable and unstable

eigenvalues,

- there are saddle points with jacobian >1 and <1 homoclinically related with p_f,

Then C1-generic g have generic dynamics, and have uncountably many chain recurrece classes which are adding machined.

Scheme of proof:

- check that we can create periodic points with derivative= identity by arbitrarily small perturbations.
- Then periodic discs with the identity as a return map
- then get larger period periodic discs by perturbation of the identity map with an arbitrar return map.
- Thus among these discs some of them carry also a universal dynamics, leading to nested sequences and uncountably many adding machines.

We have seen an example of such an open set, using Plykin attractor.
This is nice for doing an example.

But in fact such open sets are very common.

- Any cycle between saddles p,q of different indices leads to a robust cycle.

Thus for avoiding universal dynamics, the diffeomorphism f needs to have some hyperbolic structure. These leads to my attempt for a conjectural cartography (mapping) of the space $Diff^1(M)$, where the simplest are Morse-Smale diffeomorphisms and the most complicated are the universal dynamics.

Bonatti, C. Survey: Towards a global view of dynamical systems, for the C^1-topology. *Ergodic Theory Dynam. Systems* 31 (2011), no. 4, 959–993.

The idea is that there are a finite number of possibilities for weak hyperbolic/partially hyperbolic/ dominated splitting structures, and the structure which are carried on the chain recurrence class of f leads to a stratification of the diffeomorphisms f.