1 Covering Type

X space $\mathcal{U}$ cover, $\mathcal{U}$ is good, if all finite intersections are contractible

Examples
- convex geodesic balls in Riemannian manifold
- open stars of vertices in triangulated space

Minimal cardinality of a good cover is not homotopy invariant:

$sct(X)$

(Karoubi – Weibel, 2016)

$ct(X) :=$ min cardinality of good cover over all spaces homotopy equivalent to X

$X \simeq Y \Rightarrow ct(X) = ct(Y)$

$\mathcal{U}$ open cover of X with subordinated partition of unity $\{ \rho_U : X \to [0,1] \mid U \in \mathcal{U} \}$

$\rightsquigarrow$ Aleksandrov map $f : X \to |N(\mathcal{U})| \subseteq \mathbb{R}^{|\mathcal{U}|}$, $f(x) := \sum \rho_U(x) \cdot \vec{e}_U$

Nerve Theorem

$\mathcal{U}$ good cover of X with subordinated partition of unity

$\rightsquigarrow$ $f : X \to |N(\mathcal{U})|$ is a homotopy equivalence

i.e. $N(\mathcal{U})$ is homotopy triangulation of X

$\rightsquigarrow$ (for X paracompact)

$ct(X) =$ minimal number of vertices in a homotopy triangulation of X

X triangulable $\Rightarrow$

$ct(X) \leq \Delta(X)$:= minimal number of vertices in a triangulation of X

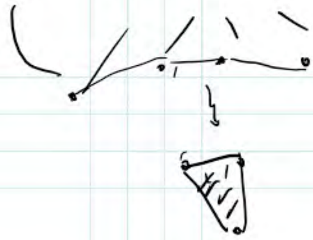
If M closed, connected manifold, can

$ct(X) \leq \Delta(X)$:= minimal number of vertices in a triangulation of X

If M closed, connected manifold, can $\Delta(M) - ct(M)$ be arbitrarily large

S closed surface (Borghini–Minian, "covering type and surfaces")

$ct(S) = \Delta(S)$ except for S = double torus (where the difference is one)



$ct(X)$ can be estimated with LS-category

e.g.: $cat(X) = cat(\bigcup \mathcal{U}) = n$

$\rightsquigarrow$ $hdim(X) \geq n-1$

$\rightsquigarrow$ $\exists\, U_{i_1}, \dots, U_{i_n} \in \mathcal{U}: U_{i_1} \cap \dots \cap U_{i_n} = \emptyset$

$\rightsquigarrow$ $U_{i_1} \cup \dots \cup U_{i_n} \simeq \bullet$ (by Nerve Thm) $\rightsquigarrow$ categorical

$\rightsquigarrow$ $\mathcal{U}' := \mathcal{U} - \{U_{i_1}, \dots, U_{i_n}\}$, $cat(\bigcup \mathcal{U}') \geq n-1$

$\rightsquigarrow$ $\exists\,(n-1)$ sets in $\mathcal{U}'$ with non-empty intersection

. . . .

X surface $\neq S^2$ $\rightsquigarrow cat X = 3$

$\rightsquigarrow$ $|\mathcal{U}| \geq n + (n-1) + \dots + 1 = \frac{n(n+1)}{2}$

$ct(\text{surface}) \geq 6$

$\rightsquigarrow$ $ct(X) \geq \frac{cat(X)\,(cat(X)+1)}{2}$

Similar approach yields estimates with cohomology products.

Variants: weighted cohomology estimates, equivariant ct, ...
computations for surfaces, Grassmannians, Lie groups, spherical forms, ...

$\boxed{\text{OPEN QUESTIONS}}$

(1) Wedges: $ct(X \vee Y)$ simple-minded upper estimates, but

is $ct(X \vee Y) \geq ct(X), ct(Y)$?

(2) Domination:

X homotopy dominates Y if there are maps

$Y \xrightarrow{f} X \xrightarrow{g} Y$ s.t. $g \circ f \simeq 1_Y$

(if f is inclusion, we also say Y is homotopy retract of X)

Homotopy invariants are often monotonous wrt to h. domination (e.g. if X h. dominates Y, then $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$, $TC(X) \geq TC(Y)$)

Natural question:

Is it true, that X h. dominates $Y \Rightarrow ct(X) \geq ct(Y)$?

Unfortunately, the answer is No.

> Fact: there exists X, s.t. $X \not\simeq$ finite CWcx but $X \times S^1 \simeq$ finite cx
>
> Then $X \times S^1$ h. dominates X, $ct(X \times S^1) < \infty$ but $ct(X) = \infty$
>
> More details: X is finitely dominated if it is h. dominated by a finite CWcx
>
> Theorem (Mather) X is fin. dominated $\Leftrightarrow X \times S^1 \simeq$ finite cx
>
> C.T.C. Wall constructed a finiteness obstruction $\sigma_X \in K_0(\mathbb{Z}(\pi_1 X))$,
>
> Theorem (Wall)
>
> (1) $\sigma_X = 0 \Leftrightarrow X \simeq$ finite cx
>
> (2) For every $\sigma \in K_0(\mathbb{Z}\pi)$, π finitely presented, exists X, s.t. $\pi_1 X = \pi$ and $\sigma_X = \sigma$.

• X, Y finite CWcx, X h. dominates $Y \implies ct(X) \geq ct(Y)$?

Dejan: is there a finite complex X that dominates infinitely many finite complexes Y_i, $i = 1, 2, \ldots$?

If so, then there would exist Y_i, s.t. $ct(Y_i) > ct(X)$

In fact, there would be infinitely many such examples and the difference $ct(Y_i) > ct(X)$ would be arbitrarily big.

This question was asked by Borsuk in Theory of Shape (Problem 6.4) and was answered in positive by

Danuta Kołodziejczyk, 2-dimensional polyhedra with infinitely many left neighbors, Top. Appl. 159 (2012), 1943 – 1947.

infinitely many left neighbors, Top. Appl. [illegible]

Explicitly: $\mathcal{P}_i := \langle r, s, t, u \mid r^2 = s^3,\ t^2 = u^3,\ r^{2i+1} = t^{2i+1},\ s^{2i+1} = u^{2i+1} \rangle$

All presentations give the trefoil group $T = \langle r, s \mid r^2 = s^3 \rangle$ but the corresponding 2-complexes $X_i = X(\mathcal{P}_i)$ have different $\pi_2(X_i)$.

However $X_i \vee \bigvee_8 S^2$ are all homotopy equivalent (by a thm of Whitehead)

Some may be equivalent after wedging with a smaller number of spheres, but there must exist an infinite family of $\{Y_j\}$, such that Y_j are pairwise non-equivalent and all $Y_j \vee S^2$ are homotopy equivalent to some finite 2-complex Y. Since there are only finitely many homotopy types of complexes with ct $\leq ct(Y)$, there are infinitely many finite 2-complexes Y_i s.t. $ct(Y_i) > ct(Y_i \vee S^1)$.

- X polyhedron, $A \leq X$ retract $\Rightarrow \Delta(X) \geq \Delta(A)$?

  (even, is $\Delta(X \times S^1) > \Delta(X)$?)

(3) Refining LS category:

$ct(X)$ increases (at least) as $cat(X)^2$

For $n > 0$ there are infinitely many homotopy types X: $cat(X) \leq n$,

(e.g., $\dim X \leq n-1 \Rightarrow cat(X) \leq n$)

but only finitely many, s.t. $ct(X) \leq n$

(homotopy types of subcomplexes of Δ_{n-1})

+ ?

Under what assumptions $cat(X) \geq cat(Y) \Rightarrow ct(X) \geq ct(Y)$

surface $cat = 3 < cat(S^1 \times S^1 \times S^1) = 4$