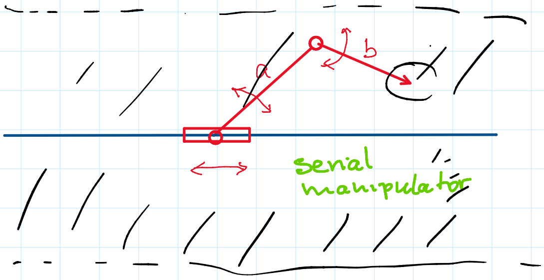
2. NICE SETS FOR MOTION PLANS

Recall basic definitions:

X configuration space of a robot or some mechanical device

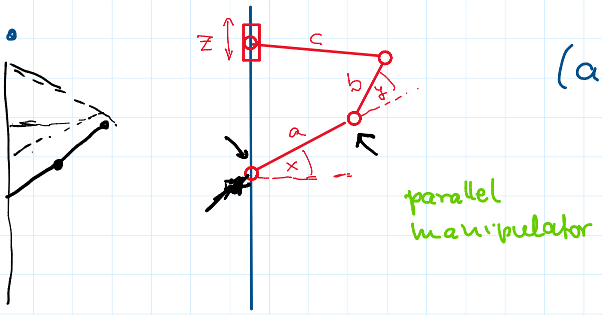
Examples

- 

Joints configuration space $J = \mathbb{R} \times S^1 \times S^1$

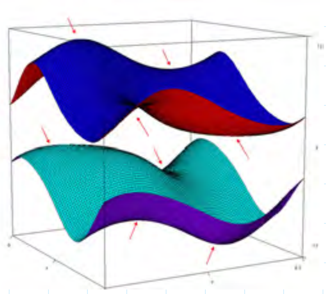
Work space $W = \mathbb{R} \times [-(a+b), (a+b)]$

$J \longrightarrow W$ kinematic map

- 

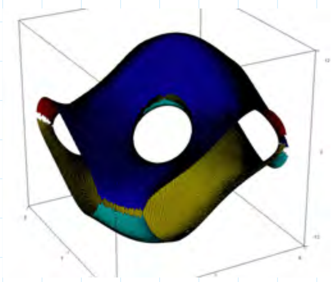
$$(a \cos x + b \cos(x+y))^2 + (a \sin x + b \sin(x+y) - z)^2 = c^2$$

Topology of the configuration space depends on the lengths a, b, c



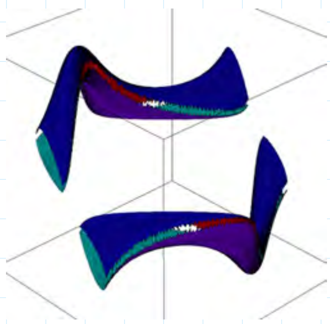
$c > a+b$

$\rightsquigarrow$ two tori



$a+b > c > a-b$

genus 3 torus



$c < a-b$

In applications one often wants to avoid singularities which further complicate the topology of the configuration spaces.

Motion planning in X: given two points x_0, x_1 in same component of X, find a path $\alpha(x_0, x_1)$ from x_0 to x_1 in X

In other words, find a section

$$\begin{tikzcd} & X^I \arrow[d, "\pi"] \\ & X\times X \arrow[u, bend left, dashed, "\alpha"] \end{tikzcd} \qquad \pi(\alpha) = (\alpha(0), \alpha(1))$$

Additional requirement, choice of paths should be robust, in the sense that α is continuous

Standard fact: continuous section $\alpha: X\times X \to X^I$ exists $\Leftrightarrow$ X is contractible

Thus, we look for subsets $U \subseteq X\times X$ that admit a section $\alpha_U : U \longrightarrow X^I$

Minimal number of partial sections needed to cover $X\times X$ is topological complexity of X, $TC(X)$

Basic properties:

(1) $X \simeq Y \Rightarrow TC(X) = TC(Y)$

(2) $TC(X) = 1 \Leftrightarrow X \simeq \bullet$ (reduced convention: $TC(\bullet) = 0$)

(3) $cat(X) \leq TC(X) \leq cat(X\times X)$

(X top. group $\Rightarrow TC(X) = cat(X)$)

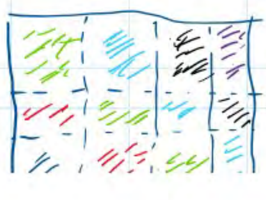
(4) $TC(X\times Y) \leq TC(X) + TC(Y) - 1$

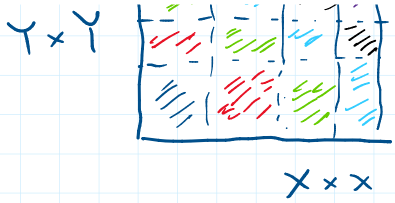
(5) $TC(X) \geq nil\left(Ker\, \Delta^* : H^*(X\times X) \longrightarrow H^*(X) \right)$

Individual motion plans $\alpha_U : U \to X^I$ are usually interpreted as different algorithms that need to be implemented. However, sometimes domain U consists of many disjoint subsets and each can be considered as a separate rule.

E.g. in CWcxes one can take interiors of n-cells, then interiors of $(n-1)$-cells and so on

For a product we estimate $TC(X\times Y)$ using "checkerboard"

$Y\times Y$ 



In view of applications we want to minimize the number (or effective complexity) of motion plans.

<u>Various approaches:</u>

- contractible is preferable to categorical as it gives more control over the motion paths.

- we also favor plans where $\alpha(x,x)$ is constant and
  $$\alpha(x,x') = \overline{\alpha(x',x)}$$

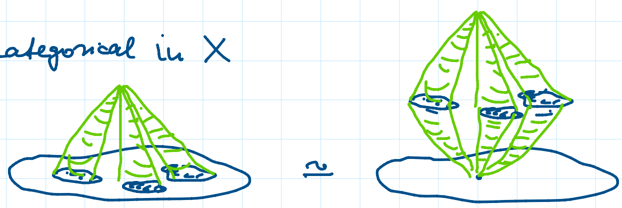
- one can split $X \times X$ into contractible pieces (e.g. interiors of cells or simplices, contract to the respective "centres" and then connect those centres to one point (preferably by a tree).

  In the case of simplices there is a bonus that we can take shortest paths

- Another way to cover a space with contractible subsets is the following:

$X = U_1 \cup \ldots \cup U_n \qquad U_i$ categorical in X

$X \cup CU_1 \simeq X \vee \Sigma U_1$



$X \vee \Sigma U_1 \cup CU_2 \simeq X \vee \Sigma(U_1 \cup U_2)$

$\ldots$

$$X \cup CU_1 \cup \ldots \cup CU_n \simeq X \vee \Sigma(\underbrace{U_1 \cup \ldots \cup U_n}_{= Z})$$

$$X \vee \Sigma Z \cup C\Sigma Z \simeq X$$

$\rightsquigarrow$ a space homotopy equivalent to X can be covered by $\mathrm{cat}X+1$ contractible sets

Drawback: such a covering requires modification of X up to homotopy

Otherwise the number of contractible subsets required to cover X

$(=\mathrm{gcat}(X))$ can be arbitrarily bigger than X.

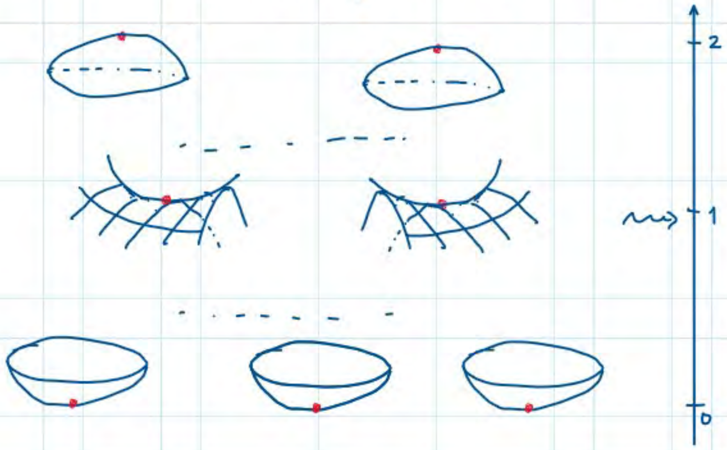
- On smooth manifolds $\rightsquigarrow$ geodesic complexity (cut-locus)

  $\rightsquigarrow$ Stephan's talk

- M smooth m-manifold can be covered by $m+1$ subspaces $\approx B^m$

Sketch of the proof:

- choose self-indexing Morse function on M



For $i=0,1,\dots,m$ $[f<i+\varepsilon]=$

$[f<i-\varepsilon]+$ disjoint index i handles

$[f\leq\frac{1}{2}]=$ disjoint union of balls

Can be enlarged in M to make ball B_0

$[f\leq\frac{3}{2}]$ isotopic $[f\leq\frac{1}{2}]+$ disjoint union of 1-handles

Enlarge in M to a ball B_1, and let $B_0:=$ image of previous B_0 under isotopy

(so that $[f\leq\frac{3}{2}]\subseteq B_0\cup B_1$)

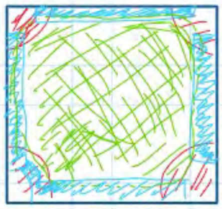
Continue, and eventually cover M with at most $m+1$ m-balls.

In particular, motion planning on M can be solved by $2m+1$ motion plans, each of them having a domain $U\subseteq M\times M$ diffeomorphic to B^{2m}.

This a good but still half-baked result.

Problems

- Explicit description of balls and respective motion plans



decomposition for $S^1\times S^1$

- Analogous version for discrete Morse function.
- Can it be done algorithmically?

Frank: given a discrete Morse function on M one may consider the Hasse diagram of critical cells and change indexing top $\to$ down to obtain a self-indexing Morse function with the same critical simplices.

- Can the balls be chosen to be collapsible?
- If M 1-connected, are $m+1$ balls enough?

Iwase-Sakai: TC is fibrewise LScat

What are fibrewise gcat, Cat and ballcat?

TC of a map

$f: X \to Y$

Interpret X as joints space, Y as work space and f as kinematic map.

In practice, f is never a fibration and rarely admits section (i.e. inverse kinematics)

Manipulation planning: given initial state $x \in X$ and final position $y \in Y$ find joints motion $\alpha(x,y) \in X^I$ such that $\alpha(x,y)(0)=x$, $f(\alpha(x,y)(1))=y$.

$$TC(f) := sec\left(\begin{array}{c} X^I \\ \downarrow \pi_f \\ X\times Y \end{array}\right)$$

= min number of manipulation plans needed to cover all possible pairs $(x,y) \in X\times Y$

If X, Y manifolds, can the domains be balls and how many?