3. COVERING TYPE OF A GROUP PRESENTATION

Problem: compute $ct(X)$ for arbitrary 2-complex X

① Group presentations and 2-complexes

There is a well-known correspondence between group presentations and 2-dimensional CW-complexes:

$\mathcal{P} = \langle \underline{x} ; \underline{r} \rangle$ presentation

$\underline{x} = \{x_1, \ldots, x_n\}$ generators

$\underline{r} = \{r_1, \ldots, r_m\}$ relations; each r_i is a word in the letters $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$

$G_{\mathcal{P}} := F(\underline{x}) / N(\underline{r})$, where

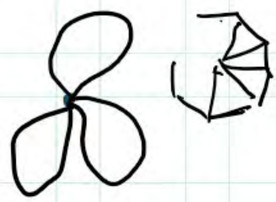
$F(\underline{x})$ = free group with generators $x_1, \ldots, x_n$

$N(\underline{r})$ = normal subgroup of $F(\underline{x})$ generated by $r_1, \ldots, r_m$

$X_{\mathcal{P}} := \underbrace{e^0 \cup \{e^1_1, \ldots, e^1_n\}}_{X^{(1)}_{\mathcal{P}}} \cup \{e^2_1, \ldots, e^2_m\}$ 2-dimensional CWcx

attaching map of e^2_i is a map $f_i : S^1 \longrightarrow X^{(1)}_{\mathcal{P}}$ corresponding to r_i

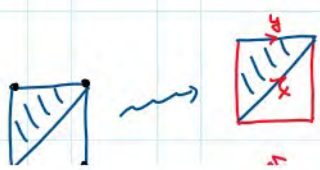
Seifert-VanKampen $\Rightarrow$ $\pi_1(X_{\mathcal{P}}, e^0) \cong G_{\mathcal{P}}$



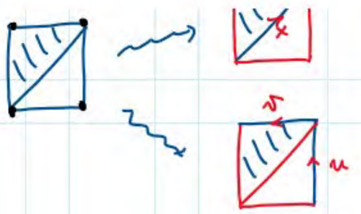
Conversely, if X is any finite 2-dim CWcx, then we can choose a max tree $T \subseteq X^{(1)}$ and form a presentation $\mathcal{P}$ where $\underline{x}$ = 1-cells in X/T, $\underline{r}$ = elements of $F(\underline{x}) \cong \pi_1(X^{(1)})$ given by attaching maps of 2-cells

Clearly $X \simeq X_{\mathcal{P}}$

Note that $\mathcal{P}$ depends on the choice of T and of orientations of 1-cells



$\mathcal{P} = \langle x, y \mid x y^{-1} \rangle$



$\mathcal{P} = \langle x, y \mid x y^{-1} \rangle$

$\mathcal{P}' = \langle u, v \mid v \rangle$

$\mathcal{P} = \langle \underline{x} \mid \underline{r} \rangle \rightsquigarrow G_{\mathcal{P}}$ and $X_{\mathcal{P}}$

However: $G_{\mathcal{P}} \cong G_{\mathcal{P}'} \not\Rightarrow X_{\mathcal{P}} \simeq X_{\mathcal{P}'}$ (easy, because $\chi(X_{\mathcal{P}}) = 1 - |\underline{x}| + |\underline{r}|$)

harder $G_{\mathcal{P}} \cong G_{\mathcal{P}'}$ & $\chi(X_{\mathcal{P}}) = \chi(X_{\mathcal{P}'}) \not\Rightarrow X_{\mathcal{P}} \simeq X_{\mathcal{P}'}$

(first examples in mid-1970's)

So, what does (homotopy type of) X determine?

in other words, what is $ct(\mathcal{P}) := ct(X_{\mathcal{P}})$?

$X \rightsquigarrow \mathcal{P}$ depends on the choice of max tree, order and orientation of generators

elementary collapses/expansions of X by 3-cells ($X \overset{3}{\searrow} Y$)

yield further modifications of $\mathcal{P}$

(1) $x_i \rightsquigarrow x_i^{-1}$
$\quad\;\; \rightsquigarrow x_i x_j \quad i \neq j$

(2) $r_i \longrightarrow r_i^{-1}$
$\quad\;\; \rightsquigarrow r_i r_j \quad i \neq j$

(3) $r_i \rightsquigarrow w r_i w^{-1}$

(4) add / remove x and $r = x$

These operations determine a presentation class of $\mathcal{P}$, denoted $[\mathcal{P}]$

THEOREM

$\mathcal{P} \mapsto X_{\mathcal{P}}$ defines a bijection between 3-deformation types of finite 2-complexes and presentation classes of finite presentations.

Thus $ct(\mathcal{P})$ is an invariant of presentation classes

If we add only one more modification $\langle \underline{x} \mid \underline{r} \rangle \rightsquigarrow \langle \underline{x} \mid \underline{r} \cup \{1\} \rangle$, then we get all Tietze transformations. $X_{\mathcal{P}} \rightsquigarrow X_{\mathcal{P}} \vee S^2$

Tietze theorem: $G_{\mathcal{P}} \cong G_{\mathcal{P}'} \Rightarrow \mathcal{P}$ can be transformed to $\mathcal{P}'$ by a sequence of Tietze transformations

Note that $\underline{r} \rightsquigarrow \underline{r} \cup \{1\}$ increases $\chi(X_{\mathcal{P}})$ by 1

Topological version: $G_{\mathcal{P}} \cong G_{\mathcal{P}'} \Rightarrow \exists k, l: X_{\mathcal{P}} \vee k S^2 \simeq X_{\mathcal{P}'} \vee l S^2$

Conjecture: $G_{\mathcal{P}} \cong G_{\mathcal{P}'}$, $\chi(X_{\mathcal{P}}) = \chi(X_{\mathcal{P}'})$ and $ct(\mathcal{P}) = ct(\mathcal{P}') \Rightarrow X_{\mathcal{P}} \simeq X_{\mathcal{P}'}$

(2) Fundamental sequence of a 2-complex

X 2-complex $\rightsquigarrow$ homotopy exact sequence of $(X, X^{(1)})$:

$$\Pi(X): 0 \to \pi_2(X) \xrightarrow{j_\#} \pi_2(X, X^{(1)}) \xrightarrow{\partial} \pi_1(X^{(1)}) \xrightarrow{i_\#} \pi_1 X \to 1$$

$$0 \to \pi_k(X) \xrightarrow{\cong} \pi_k(X, X^{(1)}) \to 0 \quad \text{for } k \geq 3$$

Only $\Pi(X)$ are relevant for us, because of

Theorem (Whitehead)

X, Y 2-cxes

$f: X \to Y$ is a homotopy equivalence $\Leftrightarrow$ $\pi_1(f)$ and $\pi_2(f)$ are isomorphisms

In more detail:

$$0 \to \pi_2(X) \xrightarrow{j_\#} \pi_2(X, X^{(1)}) \xrightarrow{\partial} \pi_1(X^{(1)}) \xrightarrow{i_\#} \pi_1 X \to 1$$

free

$\pi_1(X^{(1)})$ acts on all groups in the sequence (by basepoint translation)

+ all maps are $\pi_1(X^{(1)})$-equivariant

in particular, for $g \in \pi_1 X^{(1)}$, $c \in \pi_2(X, X^{(1)})$ $\partial(g \cdot c) = g \cdot \partial c \cdot g^{-1}$

+ for $c, d \in \pi_2(X, X^{(1)})$ $\partial c \cdot d = c \cdot d \cdot c^{-1}$ (try to check it for yourself)

We will try to give algebraic interpretation for $X = X_{\mathcal{P}}$

Algebraic abstraction:

Crossed G-module is (C, ∂, G) where (CM0) C, G groups, $\partial : C \to G$ homomorphism

(CM1) C has G-action, ∂ is equivariant

(CM2) $\partial c \cdot d = c \cdot d \cdot c^{-1}$ $(c, d \in C)$

morphism $(\eta, \tau) : (C, \partial, G) \to (C', \partial', G')$

$$\begin{tikzcd} C \arrow[r, "\partial"] \arrow[d, "\eta"'] & G \arrow[d, "\tau"] \\ C' \arrow[r, "\partial'"'] & G' \end{tikzcd}$$

commutes + η is τ-equivariant

$(\eta(g \cdot c) = \tau(g) \cdot \eta(c))$

Examples

(1) $\partial : \pi_2(X, X^{(1)}) \to \pi_1(X^{(1)})$ for X CWcx

(2) If C is a normal subgroup of G, then $i : C \hookrightarrow G$ is a crossed module

(3) $\mathcal{P} = \langle \underline{x} ; \underline{r} \rangle$ presentation

$G := F(\underline{x})$ free group on $\underline{x}$

$E(\mathcal{P}) := $ free group on the set $F(\underline{x}) \times \underline{r}$

with G-action $v \cdot (w, r) := (vw, r)$

$\partial : E(\mathcal{P}) \longrightarrow F(\underline{x}), \quad \partial(w, r) := w \cdot r \cdot w^{-1}$

note: $\operatorname{Im} \partial = N(\underline{r})$

What is $\operatorname{Ker} \partial$?

Say we have relations $y = x^2$, $z = xy$ and $z = x^3$. Clearly $z = x^3$ is implied by the first two. Thus if $\underline{r} = \{x^2y^{-1}, xyz^{-1}, x^3z^{-1}\}$,

then $x(x^2y^{-1})x^{-1} \cdot xyz^{-1} \cdot (x^3z^{-1})^{-1} = 1$ in $F(\{x, y, z\})$,

or $\partial\left((x, x^2y^{-1}) \cdot (1, xyz^{-1}) \cdot (1, x^3z^{-1})^{-1}\right) = 1$

$\operatorname{Ker} \partial$ can be interpreted as relations (='identities') between relations.

$I(\mathcal{P}) := \operatorname{Ker} \partial$ identities for the presentation $\mathcal{P}$

$\partial : E(\mathcal{P}) \to F(\underline{x})$ is not a crossed module, because

$$\partial(w, r) \cdot (v, s) = (wrw^{-1}v, s) \neq (w, r) \cdot (v, s) \cdot (w, r)^{-1}$$

basic Peiffer identity in $E(\mathcal{P})$: $(wrw^{-1}v, s) = (w, r) \cdot (v, s) \cdot (w, r)^{-1}$

Peifer subgroup $P(\mathcal{P})$: normal subgroup of $E(\mathcal{P})$ generated by basic identities

$\rightsquigarrow$ $\boxed{C(\mathcal{P}) := E(\mathcal{P})/P(\mathcal{P}) \xrightarrow{\partial} F(\underline{x})}$ is a crossed module $\Pi(\mathcal{P})$

Theorem

$C(\mathcal{P}) \xrightarrow{\partial} F(\underline{x})$ is isomorphic to $\pi_2(X_{\mathcal{P}}, X_{\mathcal{P}}^{(1)}) \xrightarrow{\partial} \pi_1(X_{\mathcal{P}}^{(1)})$

($\Pi(\mathcal{P})$ is isomorphic to $\Pi(X_{\mathcal{P}})$)

$$\begin{tikzcd} C(\mathcal{P}) \arrow[r, "\partial"] \arrow[d, "\eta"'] & F(\underline{x}) \arrow[d, "\tau"] \\ \pi_2(X_{\mathcal{P}}, X_{\mathcal{P}}^{(1)}) \arrow[r, "\partial"] & \pi_1(X_{\mathcal{P}}^{(1)}) \end{tikzcd}$$

$\eta : E(\mathcal{P}) \longrightarrow \pi_2(X_{\mathcal{P}}, X_{\mathcal{P}}^{(1)})$

$(1, r) \longmapsto \varphi_r : (B^2, S^1) \longrightarrow (X_{\mathcal{P}}, X_{\mathcal{P}}^{(1)})$

characteristic map of e_r^2.

+ extend by freeness and equivariance

$\pi_2(X_\mathcal{P}, X_\mathcal{P}^1) \to \pi_1(X_\mathcal{P}^1)$ characteristic map of e_r.

+ extend by freeness and equivariance

Corollary (Reidemeister) $\pi_2(X_\mathcal{P}) \cong I(\mathcal{P})/P(\mathcal{P})$

Note that:

1. $\Pi(\mathcal{P})$ (and thus $\Pi(X_\mathcal{P})$) is a free crossed module with base $\{(1,r) ; r \in \underline{r}\}$

2. Conversely, every free crossed module $\boxed{\text{over a free group}}$ can be realized as $\Pi(X)$ for some 2-complex X

$(\eta, \tau) : (C, \partial, G) \to (C', \partial', G')$ induces

$$\begin{tikzcd} \operatorname{Ker}\partial \arrow[r] \arrow[d, dashed, "\bar{\eta}"'] & C \arrow[r, "\partial"] \arrow[d, "\eta"] & G \arrow[r] \arrow[d, "\tau"] & \operatorname{Coker}\partial \arrow[d, dashed, "\bar{\tau}"] \\ \operatorname{Ker}\partial' \arrow[r] & C' \arrow[r, "\partial'"'] & G' \arrow[r] & \operatorname{Coker}\partial' \end{tikzcd}$$

$(\eta, \tau) : (C, \partial, G) \to (C', \partial', G')$ is **equivalence of crossed modules** if $\bar{\eta}, \bar{\tau}$ are isomorphisms

THEOREM (Whitehead)
Homotopy classification of 2-complexes coincides with classification up to equivalence of free crossed modules over free groups.

$\underset{\text{↳dominate}}{X \geq Y} \overset{?}{\Rightarrow} ct X \geq ct Y$ in particular $ct(X \vee Y) \geq ct(X)$?

(Dejan) a way toward counterexample X finite cx which dominates infinitely many different finite cxes $Y \to X \to Y$, id

Borsuk ⇝ Danuta K.

$\mathcal{P}_k = \langle x, y, u, v \mid x^2 = y^3,\ x^{2k+1} = u^{2k+1},\ y^{2k+1} = v^{2k+1} \rangle$

$\langle x, y \mid x^2 = y^3 \rangle$ trefoil, B_3

$X_k = X(\mathcal{P}_k)$ $\pi_2(X_k)$ are different

but $X_k \vee \bigvee_8 S^2$ have all the same homotopy type.

⇝ $\exists\ Y_1, Y_2, \dots$ homotopy different, but $Y_1 \vee S^2 \simeq Y_2 \vee S^2 \simeq \dots$

$\rightsquigarrow \exists\ Y_1, Y_2, \dots$ homotopy different, but

$$Y_i \vee S^2 \simeq Y_j \vee S^2 =: Y$$

$Y \geq Y_k$ $\quad ct(Y)$ for $k \gg 0$

$ct(Y_k) > ct(Y)$ $\qquad ct(Y \times S^1) < ct(Y)$

$\Delta(Y \times S^1) < \Delta(Y)$?

$A \subseteq X$ homotopy retract for some map $\Rightarrow$ $A \leq ct(X)$

3. Connection with group (co)homology

Free crossed modules are complicated objects, so it is more convenient to consider universal covers and their cellular chain complexes.

X 2-complex $\rightsquigarrow$ $\tilde{X}$ 1-connected 2-complex with free cellular $\pi_1 X$-action

Hurewicz: $\pi_2 X \cong \pi_2 \tilde{X} \cong H_2 \tilde{X}$ as $\pi_1 X$-modules

$$ct(X) = ct_{\pi_1 X}(\tilde{X})$$

Moreover $H_2(G)$ is a quotient of $H_2 X$ for every 2-complex X with $\pi_1 X \cong G$

derive lower bounds for $ct(X)$ from $H_2(\pi_1 X)$

known results for standard presentations of free groups and surface groups

compute ct for standard presentations of cyclic groups and finite abelian groups

estimate $ct(M)$ for 3-mfd M from ct of its spine