Geodesic complexity and decompositions of cut loci

Research Group Workshop on Some Problems of Applied and Computational Topology
Banach Center, Będlewo, 5-11 March 2023

Stephan Mescher
Martin Luther University
Halle-Wittenberg

Maximilian Stegemeyer
MPI for Mathematics in the Sciences
and University of Leipzig

talk based on:

S. Mescher, M. Stegemeyer, Geodesic complexity of homogeneous Riemannian manifolds, to appear in Algebr. Geom. Topol., arXiv:2105.09215

S. Mescher, M. Stegemeyer, Geodesic complexity via fibered decompositions of cut loci, J. Appl. and Comput. Topology (2022), arXiv:2206.07691

Topology and robot motion planning

Real-world situation

A robot is supposed to move autonomously from one location to another in its workspace (e.g. warehouse, grid network, ...).

Topological motion planning problem

Let X be a path-connected topological space. Given $x, y \in X$, find a path $\gamma \in PX = C^0([0,1], X)$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Definition

Let X be a top. space, $A \subset X \times X$. A motion planner over A is a map

$$s : A \to PX,$$

such that $(s(x,y))(0) = x$, $(s(x,y))(1) = y$, for all $(x,y) \in A$, i.e. a section over A of the fibration

$$\pi : PX \to X \times X, \qquad \gamma \mapsto (\gamma(0), \gamma(1)).$$

For a robot to move autonomously in X, we need a motion planner over $X \times X$.

We want robots to move *predictably*, so we want motion planners to be continuous on large subsets of $X \times X$ and only have few "jumps" in the path assignments.

Idea: Search for the lowest number of "jumps" of a motion planner that is necessary by the topology of the space.

Definition (Farber '03)

Let X be a path-connected top. space. The topological complexity of X is given by $\mathrm{TC}(X) \in \mathbb{N} \cup \{+\infty\}$,

$$\mathrm{TC}(X) := \inf\Big\{n \in \mathbb{N} \;\Big|\; \exists \bigsqcup_{j=1}^{n} A_j = X \times X,\ \text{s.t. } A_j \text{ locally compact}, \\ \forall j\ \exists s_j : A_j \to PX \text{ cont. motion planner}\Big\}.$$

$\mathrm{TC}(X)$ is a homotopy invariant and satisfies

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X),$$

where cat denotes Lusternik-Schnirelmann category.

Definition (A. Schwarz, '61)

Let $p : E \to B$ be a fibration. The sectional category or Schwarz genus of p is given by

$$\mathrm{secat}(p) = \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = B \ \text{ open cover}, s_j : U_j \overset{C^0}{\to} E, \ \ p \circ s_j = \mathrm{incl}_{U_j} \ \ \forall j \Big\}.$$

If X is an ENR (e.g. a locally finite CW complex), then:

$$\mathrm{TC}(X) = \mathrm{secat}\Big(\pi : PX \to X \times X, \ \gamma \mapsto (\gamma(0), \gamma(1)) \Big).$$

Use results by Schwarz to derive upper and lower bounds on TC(X). Lower bounds are mostly obtained from studying the cohomology rings of X.

Motivation for geodesic complexity

(after David Recio-Mitter, 2020)

Problem Motion planners with few domains of continuity may consist of paths that are not feasible or very inefficient.

Engineers might prefer *efficient* motion along short paths and put up with discontinuities.

Idea Given a geodesic space (X, d), i.e. a metric space in which any two points are connected by a minimal geodesic, we allow only paths having minimal length.

Definition (Recio-Mitter 2020) (M,g) complete Riemannian manifold, $GM := \{\text{minimal geodesics in } M\} \subset PM$, $\pi : GM \to M \times M$, $\pi(\gamma) = (\gamma(0), \gamma(1))$, and $A \subset M \times M$. The geodesic complexity of A in M is given by

$$\mathrm{GC}_M(A) := \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigsqcup_{j=1}^{n} B_j \supset A, \text{ s.t. } B_j \text{ locally compact}, \\ \forall j\ \exists s_j : B_j \xrightarrow{C^0} GM \text{ with } \pi \circ s_j = \mathrm{incl}_{B_j} \Big\}.$$

Put $\mathrm{GC}(M,g) := \mathrm{GC}_M(M \times M)$ - the geodesic complexity of M.

A local section $s : B \to GM$ of π is called a geodesic motion planner on B.

Caveat $\pi : GM \to M \times M$ is not a fibration, so can not use general results on secat etc.

Observations on geodesic complexity

- $\mathrm{GC}_M(A \cup B) \leq \mathrm{GC}_M(A) + \mathrm{GC}_M(B)$ for all $A, B \subset M \times M$.
- $\mathrm{TC}(M) \leq \mathrm{GC}(M, g)$ for every complete Riemannian mfld. (M, g).
- The standard examples of motion planners show that

$$\mathrm{GC}(S^n, g_{\text{round}}) = \mathrm{TC}(S^n) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}.$$

- GC really depends on the Riemannian metric. Shown by Recio-Mitter:

$$\mathrm{GC}(T^2, g_{\text{flat}}) = 3, \quad \mathrm{GC}(T^2, g_{\mathbb{R}^3}) = 4.$$

- **Recio-Mitter, 2020:** For every $k \in \mathbb{N}$ there exists a closed Riem. manifold (M, g) with

$$\mathrm{GC}(M, g) \geq \mathrm{TC}(M) + k.$$

Cut loci in Riemannian manifolds

Let (M, g) complete Riemannian manifold. Let $p \in M$ and let $\gamma : \mathbb{R} \to M$ be a geodesic with $\gamma(0) = p$. Put

$$t_* := \sup\{t > 0 \mid \gamma|_{[0,t]} \text{ is length-minimizing}\}.$$

- $q := \gamma(t_*)$ is called a cut point of p. The cut locus of p is

$$\mathrm{Cut}_p(M) := \{\text{cut points of } p\} \subset M.$$

- $t_* \cdot \gamma'(0) \in T_pM$ is called a tangent cut point of p. The tangent cut locus of p is

$$\widetilde{\mathrm{Cut}}_p(M) := \{\text{tangent cut points of } p\} \subset T_pM.$$

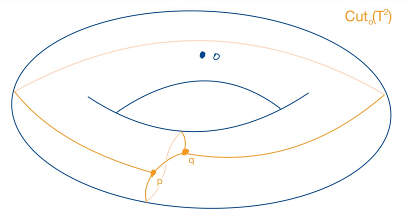
- Clearly, the Riemannian exponential map of (M, g) satisfies

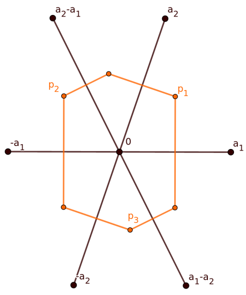
$$\exp_p\left(\widetilde{\mathrm{Cut}}_p(M)\right) = \mathrm{Cut}_p(M).$$

- If M is compact, then $\widetilde{\mathrm{Cut}}_p(M) \approx S^n$ for each $p \in M$.

Example for cut locus: flat 2-tori

Consider $T^2 = \mathbb{R}^2/\Gamma$ with quotient metric of standard metric, where $\Gamma \subset \mathbb{R}^2$ is a lattice whose generators $\{a_1, a_2\}$ satisfy $a_1 \not\perp a_2$. $o := [0]$.





$\mathrm{Cut}_o(T^2)$ $\widetilde{\mathrm{Cut}_o}(T^2)$

The trouble with geodesic motion planning on cut loci

(M, g) complete Riemannian manifold, $\mathrm{Cut}_p(M)$ the cut locus of $p \in M$.

Proposition
Let $p \in M$ and $q \in \mathrm{Cut}_p(M)$, such that there exist $\gamma_1, \gamma_2 \in GM$ from p to q with $\gamma_1 \neq \gamma_2$. Let U be an open neighborhood of q. Then there exists no continuous geodesic motion planner on $\{p\} \times U$.

Proof The map $v : GM \to TM$, $v(\gamma) = \gamma'(0)$, is continuous. If $s : \{p\} \times U \to GM$ was a cont. geod. motion planner, then $v \circ s : \{p\} \times U \to TM$ was continuous. For $i \in \{1, 2\}$ one computes:

$$\lim_{t \nearrow 1} (v \circ s)(p, \gamma_i(t)) = \lim_{t \nearrow 1} t \cdot \gamma_i'(0) = \gamma_i'(0)$$

$$\Rightarrow \quad \lim_{t \nearrow 1} (v \circ s)(p, \gamma_1(t)) \neq \lim_{t \nearrow 1} (v \circ s)(p, \gamma_2(t)) \quad \lightning$$

Contradiction, as $\lim_{t \nearrow 1} s(p, \gamma_1(t)) = s(p, q) = \lim_{t \nearrow 1} s(p, \gamma_2(t))$. $\square$

Total cut loci in Riemannian manifolds

- (Recio-Mitter, 2020) The total cut locus of M is given by

$$\mathrm{Cut}(M) := \bigcup_{p\in M} (\{p\} \times \mathrm{Cut}_p(M)) \subset M \times M.$$

- The total tangent cut locus of M is given by

$$\widetilde{\mathrm{Cut}}(M) := \bigcup_{p\in M} \widetilde{\mathrm{Cut}}_p(M) \subset TM.$$

- The extended exponential map

$$\mathrm{Exp} : TM \to M \times M, \quad \mathrm{Exp}(p, v) = (p, \exp_p(v)),$$

satisfies

$$\mathrm{Exp}\left(\widetilde{\mathrm{Cut}}(M)\right) = \mathrm{Cut}(M).$$

- If $(p, q) \notin \mathrm{Cut}(M)$, then there exists a unique minimal geodesic from p to q.

General facts about cut loci

Let (M, g) be a complete Riemannian manifold and $p \in M$.

- $\mathrm{Cut}_p(M)$ is closed and of Lebesgue measure zero.
- The set of all $q \in M$ for which $\exists \gamma_1, \gamma_2 \in GM$ with $\gamma_1 \neq \gamma_2$ joining p and q is dense in $\mathrm{Cut}_p(M)$.
- **Gluck, Singer 1978:** Every smooth manifold of dimension ≥ 2 admits a Riemannian metric with a non-triangulable cut locus.
- For $q \neq p \in M$, the sets $\mathrm{Cut}_p(M)$ and $\mathrm{Cut}_q(M)$ might be wildly different from each other.
- Let $\mathrm{Isom}(M, g) := \{\phi : M \to M \mid \phi \text{ is an isometry}\}$. If $\phi \in \mathrm{Isom}(M, g)$, then $\phi(\mathrm{Cut}_p(M)) = \mathrm{Cut}_{\phi(p)}(M)$.

Geodesic motion planners and cut loci

- For a complete Riemannian manifold (M, g) the unique geodesic motion planner

$$s : (M \times M) \setminus \mathrm{Cut}(M) \to GM, \quad (s(p,q))(t) = \exp_p(t \cdot \mathrm{Exp}^{-1}(p,q)),$$

is continuous. (see also **Blaszczyk, Carrasquel-Vera, 2018**)

- We derive:

$$\mathrm{GC}_M(\mathrm{Cut}(M)) \leq \mathrm{GC}(M,g) \leq \mathrm{GC}_M(\mathrm{Cut}(M)) + 1.$$

(Conjecture: It always holds that $\mathrm{GC}(M,g) = \mathrm{GC}_M(\mathrm{Cut}(M)) + 1$.)

- Let $A \subset \mathrm{Cut}(M)$. If $\sigma_A : A \to \widetilde{\mathrm{Cut}}(M)$ is a continuous section of $\mathrm{Exp}\,|_{\widetilde{\mathrm{Cut}}(M)} : \widetilde{\mathrm{Cut}}(M) \to \mathrm{Cut}(M)$, then

$$s_A : A \to GM, \qquad (s_A(p,q))(t) = \exp_p(t \cdot \sigma_A(p,q)),$$

is a geodesic motion planner.

- **Idea for GC** Find upper bounds on the numbers of elements of a decomposition of $\mathrm{Cut}(M)$ into domains of continuous sections of $\mathrm{Exp}\,|_{\widetilde{\mathrm{Cut}}(M)}$.

Fibered decompositions of the total cut locus

Definition Let (M, g) be a complete Riemannian manifold. A decomposition of $\mathrm{Cut}(M)$ into locally compact subsets $A_1, \dots, A_k$ is called a fibered decomposition of $\mathrm{Cut}(M)$ if, with $\widetilde{A}_i := \mathrm{Exp}^{-1}(A_i) \cap \widetilde{\mathrm{Cut}}(M)$, the restriction

$$\pi_i := \mathrm{Exp}\,|_{\widetilde{A}_i} : \widetilde{A}_i \to A_i$$

is a fibration for each $i \in \{1, 2, \dots, k\}$.

Theorem (M., Stegemeyer, 2022)

If $\mathrm{Cut}(M)$ admits a fibered decomposition $\{A_1, \dots, A_k\}$, then in the above notation

$$\mathrm{GC}(M, g) \leq \sum_{i=1}^{k} \mathrm{secat}(\pi_i : \widetilde{A}_i \to A_i) + 1.$$

$$\mathrm{GC}(M, g) \geq \max\{\mathrm{secat}(\pi_i) \mid i \in \{1, 2, \dots, k\}\}.$$

Q Which Riemannian manifolds admit such fibered decompositions?

Idea Restrict our attention to homogeneous manifolds (i.e. $\mathrm{Isom}(M, g)$ acts transitively on M), where all cut loci "look the same".

An existence result for fibered decompositions

Theorem (M., Stegemeyer, 2022)

a) If M is an irreducible[1] compact simply connected symmetric space, then $\mathrm{Cut}(M)$ admits a fibered decomposition.

b) Consider the lens space $L(p,1) \cong S^3/\mathbb{Z}_p$, with a Riemannian metric of constant curvature g. Then $\mathrm{Cut}(L(p,1))$ admits a fibered decomposition.

Strategy Let (M,g) be a homogeneous Riemannian manifold. Find a fibered decomposition of $\mathrm{Cut}_p(M)$ with an additional property and extend it to a fibered decomposition of $\mathrm{Cut}(M)$ via isometries.

[1] M is *irreducible* if it is not isometric to a product of symmetric spaces.

Isotropy-invariant decompositions

Let (M, g) be a homogenous Riemannian manifold, put $G := \mathrm{Isom}(M)$ and let $\Phi : G \times M \to M$ denote its action. For $p \in M$ let G_p denote its isotropy group.

Definition A locally compact decomposition $\{B_1, \dots, B_k\}$ of $\mathrm{Cut}_p(M)$ is called isotropy-invariant if

$$\Phi_g(B_i) \subset B_i \qquad \forall g \in G_p,\ i \in \{1, 2, \dots, k\}.$$

Proposition (M., Stegemeyer 2022) Let $p \in M$ and let $\{B_1, \dots, B_k\}$ be an isotropy-invariant decomposition of $\mathrm{Cut}_p(M)$. Put $\widetilde{B}_i := \exp_p^{-1}(B_i) \cap \widetilde{\mathrm{Cut}_p}(M)$ and assume that $\exp_p|_{\widetilde{B}_i} : \widetilde{B}_i \to B_i$ is a fibration for each $i \in \{1, 2, \dots, k\}$.

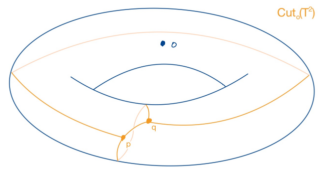
a) Then $\{A_1, \dots, A_k\}$ is a fibered decomposition of $\mathrm{Cut}(M)$, where

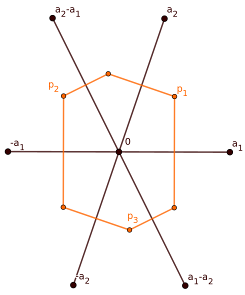
$$A_i := \{(q, r) \in M \times M \mid \exists g \in G \text{ s.t. } \Phi_g(p) = q \text{ and } r \in \Phi_g(B_i)\}.$$

b) $p_i : A_i \to M$, $p_i(q, r) = q$, is a fiber bundle with fiber B_i for each i.

Example for isotropy-invariant decomposition: flat 2-tori

$T^2 = \mathbb{R}^2/\Gamma$ with quotient metric g_Γ of standard metric, where $\Gamma = \mathbb{Z} \cdot \{a_1, a_2\}$, $a_1 \not\parallel a_2$. $o := [0]$.





$B_1 = \{p, q\}$, $B_2 = \mathrm{Cut}_o(T^2) \setminus \{p, q\}$.

$\widetilde{B}_1 = \{$vertices of the hexagon$\}$, $\widetilde{B}_2 = \{$points on the edges of the hexagon$\}$.

$\exp_o|_{\widetilde{B}_1} : \widetilde{B}_1 \to B_1$ is a trivial 3-fold covering, $\exp_o|_{\widetilde{B}_2} : \widetilde{B}_2 \to B_2$ a trivial 2-fold covering.

Example for isotropy-invariant decomposition: flat 2-tori (3)

$\{B_1, B_2\}$ is isotropy-invariant.

For the induced fibered decomposition $\{A_1, A_2\}$ of $\mathrm{Cut}(T^2)$ the fibrations

$$\pi_1 : \widetilde{A}_1 \to A_1, \qquad \pi_2 : \widetilde{A}_2 \to A_2,$$

are trivial, hence

$$\mathrm{GC}(T^2, g_\Gamma) \leq \mathrm{secat}(\pi_1) + \mathrm{secat}(\pi_2) + 1 = 1 + 1 + 1 = 3.$$

Since $\mathrm{TC}(T^2) = 3$, this shows that

$$\mathrm{GC}(T^2, g_\Gamma) = 3$$

for any such lattice Γ.

Another approach to the GC of homogeneous Riemannian manifolds

Another idea of relating cut loci of points with total cut loci in homogeneous manifolds:

Proposition If (M, g) is homogeneous and $p \in M$, then $e_p : \mathrm{Isom}(M, g) \to M$, $\phi \mapsto \phi(p)$, is a principal G-bundle, where G is the isotropy group of p.

Theorem (M., Stegemeyer 2021) If (M, g) is homogeneous, then

$$\mathrm{GC}(M, g) \leq \mathrm{secat}(e_p : \mathrm{Isom}(M, g) \to M) \cdot \mathrm{GC}_M(\mathrm{Cut}_p(M)) + 1.$$

Corollary If (G, g_{inv}) is a conn. Lie group with left-invariant Riemannian metric, then

$$\mathrm{GC}(G, g_{\mathrm{inv}}) \leq \mathrm{GC}_M(\mathrm{Cut}_1(G)) + 1.$$

Theorem (M., Stegemeyer 2021) Let $S^3 \cong SU(2)$ be equipped with a Berger metric g_B. Then $\mathrm{GC}(S^3, g_B) = 2$.

Problem The factor $\mathrm{secat}(e_p)$ is hard to compute and might become very big, only know in general that $\mathrm{secat}(e_p) \leq \mathrm{cat}(M)$.

Cut loci of irreducible compact simply connected symmetric spaces

Let $M = G/K$ be an irreducible compact simply connected symmetric space, where (G, K) is a Riemannian symmetric pair, let $o = [1] \in M$.

T. Sakai, 1978: Using root systems and properties of adjoint representations, one can show:

- $\mathrm{Cut}_o(M)$ has a decomposition $\{S_1, \ldots, S_r\}$, where $r = \mathrm{rank}(M)$.
- Each connected component $W \subset S_i$, $i \in \{1, 2, \ldots, r\}$ is of the form

$$W \approx K/Z \times B^{r-i},$$

  where B^{r-i} is an open $(r-i)$-ball, $Z \leq K$ is a closed subgroup, depending on a root system of (G, K).
- If $W_1, W_2 \subset S_i$ are connected components with $W_1 \neq W_2$, then $\overline{W}_1 \cap W_2 = \varnothing$.

Lemma $\{S_1, \ldots, S_r\}$ is isotropy-invariant and gives rise to a fibered decomposition $\{A_1, \ldots, A_r\}$ of $\mathrm{Cut}(M)$.

Theorem (M., Stegemeyer, 2022) Let $M = G/K$ be an irreducible compact simply connected symmetric space. Then, in terms of the fibered decomposition from above,

$$\begin{aligned}\mathrm{GC}(M, g_{\mathrm{sym}}) &\leq \sum_{i=1}^{r} \max\{\mathrm{secat}(\pi_i|_{\widetilde{A}_i|_W} : \widetilde{A}_i|_W \to W) \mid W \in \pi_0(A_i)\} + 1 \\ &\leq \sum_{i=1}^{r} \max\{\mathrm{cat}(W) \mid W \in \pi_0(A_i)\} + 1.\end{aligned}$$

Example $G_2(\mathbb{C}^4) = U(4)/(U(2) \times U(2))$, $\mathrm{TC}(G_2(\mathbb{C}^4)) = 9$.

Sakai, 1978: $\mathrm{Cut}_o(G_2(\mathbb{C}^4)) = S_1 \sqcup S_2$, where S_1 is a simply conn. manifold with $\dim S_1 = 6$, $S_2 \approx (S^2 \times S^2) \sqcup \{*\}$.

Hence, $\mathrm{Cut}(G_2(\mathbb{C}^4)) = A_1 \cup A_2$, where A_1 is a bundle over $G_2(\mathbb{C}^4)$ with fiber S_1, $A_2 = C_1 \cup C_2$, where C_1 is a bundle over $G_2(\mathbb{C}^4)$ with fiber $S^2 \times S^2$ and $C_2 \approx G_2(\mathbb{C}^4)$.

$$\Rightarrow \mathrm{GC}(G_2(\mathbb{C}^4)) \leq \mathrm{cat}(A_1) + \mathrm{cat}(C_1) + 1 \leq 8 + 7 + 1 = 16.$$

Some well-established facts on complex projective spaces:

- Well-known that $\mathrm{TC}(\mathbb{C}P^n) = 2n+1$ for each $n \in \mathbb{N}$.
- $\mathbb{C}P^n = U(n+1)/(U(n) \times U(1))$, $n \geq 2$, with the Fubini-Study metric g_{FS} is an irred. compact simply connected symmetric space of rank one.
- A classical result from Riemannian geometry: $\mathrm{Cut}_p(\mathbb{C}P^n) \approx \mathbb{C}P^{n-1}$, one obtains that

$$\mathbb{C}P^{n-1} \hookrightarrow \mathrm{Cut}(\mathbb{C}P^n) \to \mathbb{C}P^n$$

  is a fiber bundle.

Geodesic complexity of complex projective spaces (2)

- By the above theorem, we obtain

$$\mathrm{GC}(\mathbb{C}P^n, g_{FS}) \leq \mathrm{secat}(\mathrm{Exp} : \widetilde{\mathrm{Cut}}(\mathbb{C}P^n) \to \mathrm{Cut}(\mathbb{C}P^n)) + 1 \leq \mathrm{cat}(\mathrm{Cut}(\mathbb{C}P^n)) + 1.$$

- Since fiber and base are simply conn., $\mathrm{Cut}(\mathbb{C}P^n)$ is simply conn., hence

$$\mathrm{cat}(\mathrm{Cut}(\mathbb{C}P^n)) \leq \frac{\dim \mathrm{Cut}(\mathbb{C}P^n)}{2} + 1 = \frac{\dim \mathbb{C}P^n + \dim \mathbb{C}P^{n-1}}{2} + 1 = 2n.$$

  Thus, $\mathrm{GC}(\mathbb{C}P^n, g_{FS}) \leq 2n + 1$.
- Since $\mathrm{TC}(\mathbb{C}P^n) \leq \mathrm{GC}(\mathbb{C}P^n, g_{FS})$, we obtain

$$\mathrm{GC}(\mathbb{C}P^n, g_{FS}) = 2n + 1.$$

- Analogously, one shows that $\mathrm{TC}(\mathbb{H}P^n) = \mathrm{GC}(\mathbb{H}P^n, g_{\mathrm{sym}}) = 2n + 1$.

Let $\pi : \widetilde{M} \to M = \widetilde{M}/\Gamma$ be a Riemannian covering, Γ a finite group of isometries of $\widetilde{M}$. Given $q \in \widetilde{M}$, let

$$\Delta_q := \{r \in \widetilde{M} \,|\, d_{\widetilde{M}}(q, r) < d_{\widetilde{M}}(g \cdot q, r) \quad \forall g \in \Gamma\}.$$

Ozols, 1974: If $\overline{\Delta}_q \cap \mathrm{Cut}_q(\widetilde{M}) = \varnothing$, then $\mathrm{Cut}_{\pi(q)}(M) = \pi(\partial \Delta_q)$.

Consider the 3-sphere as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and the $\mathbb{Z}_p$-action given by $m \cdot (z_1, z_2) = (e^{\frac{2\pi i m}{p}} z_1, e^{\frac{2\pi i m}{p}} z_2)$.

For $p \geq 3$ we consider

$$L(p, 1) := S^3/\mathbb{Z}_p$$

and equip $L(p, 1)$ with the metric induced by the round metric on S^3. Then

$$\pi : S^3 \to L(p, 1)$$

is a Riemannian covering.

Farber, Grant, 2008: $\mathrm{TC}(L(p, 1)) = 6$.

(Cut loci of $L(p,1)$ have been determined by **S. Anisov, 2006**, our computations are independent.)

A long, but straightforward computation (carried out by **M. Stegemeyer**) derives from Ozols' theorem for $q \in L(p,1)$ that

$$\mathrm{Cut}_q(L(p,1)) = B_2 \sqcup B_p,$$

where

$B_2 = \{r \in L(p,1) \mid \text{there are precisely 2 minimal geodesics from } q \text{ to } r\}$,

$B_p = \{r \in L(p,1) \mid \text{there are precisely } p \text{ minimal geodesics from } q \text{ to } r\} \approx S^1$.

$\{B_2, B_p\}$ is isotropy-invariant and induces a fibered decomposition $\mathrm{Cut}(L(p,1)) = A_2 \sqcup A_p$.

$$\Rightarrow \quad \mathrm{GC}(L(p,1)) \leq \mathrm{secat}(\pi_2 : \widetilde{A}_2 \to A_2) + \mathrm{secat}(\pi_p : \widetilde{A}_p \to A_p) + 1.$$

$\pi_2 : \widetilde{A}_2 \to A_2$ is a trivial covering, hence $\mathrm{secat}(\pi_2) = 1$.
A_p is a circle bundle over $L(p,1)$, hence

$$\mathrm{secat}(\pi_p) \leq \mathrm{cat}(A_p) \leq \dim A_p + 1 = 4 + 1 = 5.$$

Thus, $\mathrm{GC}(L(p,1)) \leq 7. \quad \Rightarrow \quad \mathrm{GC}(L(p,1)) \in \{6, 7\}$.

Questions for future research

- When does a closed manifold M admit a Riemannian metric g such that $\mathrm{TC}(M) = \mathrm{GC}(M, g)$?
- In case that $\mathrm{TC}(M) = \mathrm{GC}(M, g)$: find explicit geodesic motion planners on M having $\mathrm{TC}(M)$ domains of continuity.
- How does GC behave with respect to products and Riemannian coverings?
- Are there stability properties of GC under perturbations of the Riemannian metric?
- Find more examples of Riemannian manifolds with well-understood cut loci whose GC can be determined.

Thank you for your attention!

talk based on:

S. Mescher, M. Stegemeyer, Geodesic complexity of homogeneous Riemannian manifolds, to appear in Algebr. Geom. Topol., arXiv:2105.09215

S. Mescher, M. Stegemeyer, Geodesic complexity via fibered decompositions of cut loci, J. Appl. and Comput. Topology (2022), arXiv:2206.07691

Bonus: A lower bound for stratified cut loci

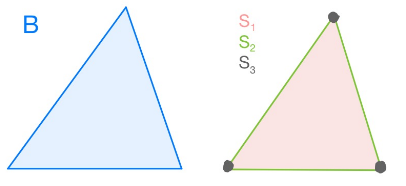
A lower bound for stratified cut loci (1)

Aim We want to derive another lower bound for GC for cut loci admitting stratifications.

Definition $B \subset M$. A *stratification of* B *of depth* N is a family $(S_1, S_2, \ldots, S_N)$, s.t.

- $S_i \subset B$ locally closed, $S_i \cap S_j = \varnothing \ \forall i \neq j$,
- $B = \bigcup_{i=1}^{n} S_i, \quad \overline{S}_i = \bigcup_{j=i}^{N} S_j \quad \forall i \in \{1, 2, \ldots, N\}$.
- Z_i conn. component of S_i, Z_j conn. component of S_j. Then:

$$Z_j \cap \overline{Z}_i \neq \varnothing \quad \Rightarrow \quad Z_j \subset \overline{Z}_i.$$



A lower bound from the structure of a cut locus

(M,g) closed Riem. manifold, $p \in M$. Let $U_p \subset T_pM$ be the domain of injectivity of $\exp_p$. Put $K := \overline{U}_p = U_p \cup \widetilde{\mathrm{Cut}}_p(M)$ and $\exp_K := \exp_p|_K : K \to M$.

Definition A stratification $(S_1, \dots, S_N)$ of $\mathrm{Cut}_p(M)$ is inconsistent if for all $i \in \{2, 3, \dots, N\}$ and $x \in S_i$ the following holds:

$\exists$ open nbhd. U of x, so that with $\pi_0(U \cap S_{i-1}) = \{Z_1, Z_2 \dots, Z_s\}$:

$$x \in \overline{Z}_j \ \ \forall j \quad \wedge \quad \widetilde{\mathrm{Cut}}_p(M) \cap \exp_p^{-1}(\{x\}) \cap \bigcap_{j=1}^{s} \overline{\exp_K^{-1}(Z_j)} = \varnothing.$$

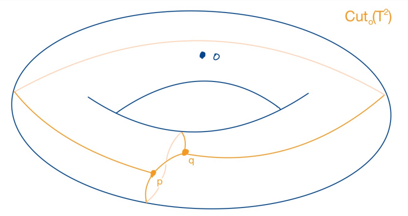
Use inconsistency of cut loci as a *geometric obstruction* to existence of continuous geodesic motion planners.

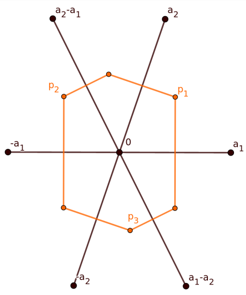
Theorem (M.-Stegemeyer '21) If $\exists p \in M$ for which $\mathrm{Cut}_p(M)$ admits an inconsistent stratification of depth N, then

$$\mathrm{GC}(M,g) \geq N+1.$$

Example of inconsistent stratification: flat 2-tori (1)

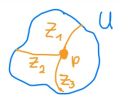
$T^2 = \mathbb{R}^2/\Gamma$ with quotient metric g_Γ of standard metric, where $\Gamma = \mathbb{Z}\cdot\{a_1, a_2\}$, $a_1 \not\parallel a_2$. $o := [0]$.





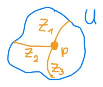
$\mathrm{Cut}_o(T^2)$ $\widetilde{\mathrm{Cut}}_o(T^2)$

Stratification (S_1, S_2) of $\mathrm{Cut}_o(T^2)$: $S_2 = \{p, q\}$, $S_1 = \mathrm{Cut}_o(T^2) \setminus \{p, q\}$. Choose a sufficiently small neighborhood U of p.

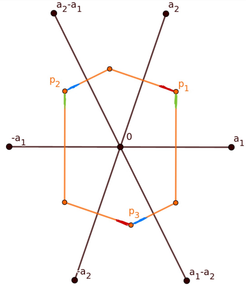


Example of inconsistent stratifications: flat 2-tori (2)

Choose a sufficiently small neighborhood U of p and let

$\pi_0(U \cap S_1) = \{Z_1, Z_2, Z_3\}$. 

$\widetilde{\text{Cut}}_o(T^2) \cap \exp_o^{-1}(Z_1)$, $\widetilde{\text{Cut}}_o(T^2) \cap \exp_o^{-1}(Z_2)$, $\widetilde{\text{Cut}}_o(T^2) \cap \exp_o^{-1}(Z_3)$.



$$\widetilde{\text{Cut}}_o(T^2) \cap \exp_o^{-1}(\{p\}) \cap \overline{\exp_K^{-1}(Z_1)} \cap \overline{\exp_K^{-1}(Z_2)} \cap \overline{\exp_K^{-1}(Z_3)}$$

$$= \{p_1, p_3\} \cap \{p_2, p_3\} \cap \{p_1, p_2\} = \varnothing. \quad \Rightarrow \quad (S_1, S_2) \text{ is inconsistent.}$$

Metrics with non-degenerate cut points

Definition (Itoh-Sakai 2007) (M, g) closed, $p \in M$, $q \in \mathrm{Cut}_p(M)$, $k \in \mathbb{N}$.

a) Call q of order $k+1$ if there are precisely $k+1$ distinct minimal geodesics $\gamma_0, \gamma_1, \ldots, \gamma_k : [0,1] \to M$ from p to q.

b) We call q non-degenerate if additionally $\{\dot\gamma_0(1), \dot\gamma_1(1), \ldots, \dot\gamma_k(1)\} \subset T_qM$ is in general position (i.e. $\{\dot\gamma_1(1) - \dot\gamma_0(1), \ldots, \dot\gamma_k(1) - \dot\gamma_0(1)\}$ is linearly independent)

Theorem

a) ***(Itoh-Sakai 2007)*** *Assume* $\mathrm{Cut}_p(M)$ *contains no conjugate point of* p *and that every* $q \in \mathrm{Cut}_p(M)$ *is non-degenerate. Then* $(C_N, C_{N-1}, \ldots, C_1)$ *is a stratification of* $\mathrm{Cut}_p(M)$*, where*

$$C_k = \{q \in \mathrm{Cut}_p(M) \mid q \text{ is of order } k+1\} \quad \forall k \in \{1, 2, \ldots, N\}$$

and N *is the highest order of a point in* $\mathrm{Cut}_p(M)$.

b) ***(M.-Stegemeyer 2021)*** *This stratification is inconsistent.*

Again: Thank you for your attention!

talk based on:

S. Mescher, M. Stegemeyer, Geodesic complexity of homogeneous Riemannian manifolds, to appear in Algebr. Geom. Topol., arXiv:2105.09215

S. Mescher, M. Stegemeyer, Geodesic complexity via fibered decompositions of cut loci, J. Appl. and Comput. Topology (2022), arXiv:2206.07691