Fundamental Groups of Small Simplicial Complexes

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Motivating example: Poincaré sphere

Let $G = 2I$ be the binary icosahedral group and $P = S^3/G$.

- Björner-Lutz (2000): P has a triangulation with 16 vertices.
- Bagchi-Datta (2005): ≥ 12 vertices needed to triangulate P.

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Conjecture (Björner-Lutz)

The minimal triangulation of P has 16 vertices.

General idea

If τ is a facet of P (resp. a d-manifold M, $d \geq 3$):

$$\pi_1(P) \cong \pi_1(P \setminus \tau) \cong \pi_1(P \setminus \bigcup_{v \in \tau} \mathrm{St}(v)).$$

General idea

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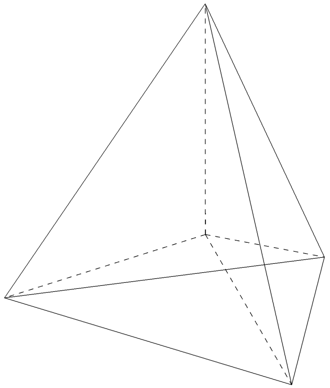
Corollary

If M (d-manifold) can be triangulated with n vertices, there is a complex X with $n - d - 1$ vertices such that $\pi_1(X) \cong \pi_1(M)$.

Observation (Björner-Lutz)

There is a complex X with 10 vertices such that $\pi_1(X) = \pi_1(P)$.

Up to homotopy, these are all (sums of) wedges of spheres:



Complexes with ≤ 5 vertices

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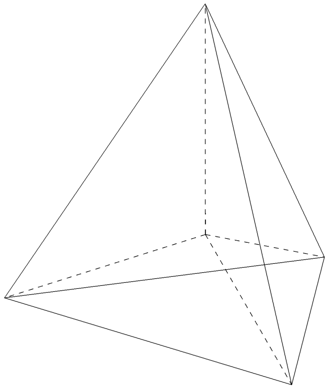
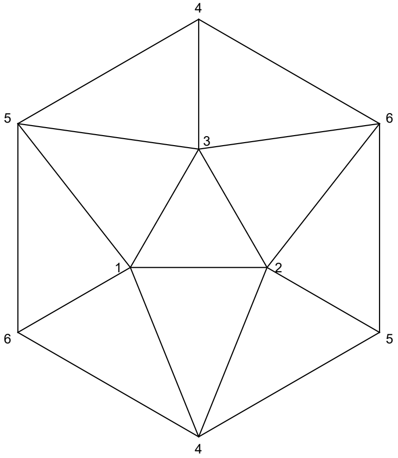


Therefore, $\pi_1(X) \cong F_n$ (free group).

With 6 vertices, $\mathbb{R}P^2$ can be realized:



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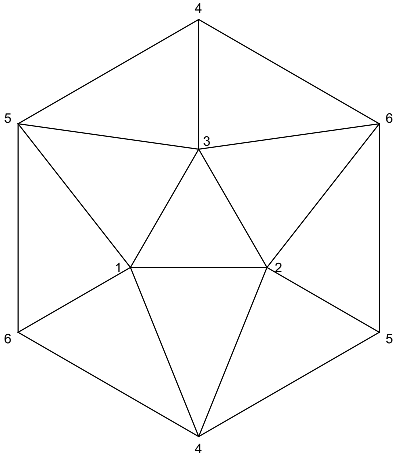


In fact, either $\pi_1(X) \cong F_n$ or $\pi_1(X) = \mathbb{Z}_2$.

Dedekind Numbers

How many complexes on 7 vertices are there?

n	d_n	r_n
0	2	2
1	3	3
2	6	5
3	20	10
4	168	30
5	7 581	210
6	7 828 354	16 353
7	2 414 682 040 998	490 013 148
8	56 130 437 228 687 557 907 788	1 392 195 548 889 993 358
9	???	???

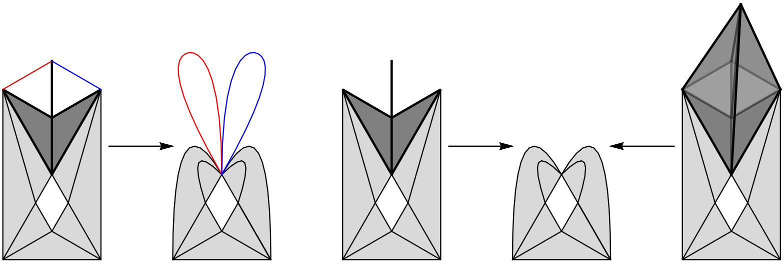
Number of 2-Pure Complexes

Observation

WLOG, X is pure 2-dimensional, since $\pi_1(X) \cong \pi_1(X^{(2)})$.

n	t_n
0	1
1	1
2	1
3	2
4	5
5	34
6	2 136
7	7 013 320
8	1 788 782 616 656
9	53 304 527 811 667 897 248

A Lemma



Complexes with ≤ 7 vertices

Procedure:

- Generate 2-pure complexes using `nauty-geng`.
- Use cone-and-collapse (in `Mathematica`) to exclude obvious wedges of spheres. This leaves 602 complexes.
- Compute π_1 for these complexes in `SageMath`.

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Result

If X is a (connected) complex on ≤ 7 vertices:

- $\pi_1(X) \cong F_n$, *where* $0 \leq n \leq 15$,
- $\pi_1(X) \cong \mathbb{Z}_2 * F_n$, *where* $0 \leq n \leq 5$, *or*
- $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$.

This reproves: ≥ 12 vertices needed to triangulate P.

What next?

On 8 vertices we have:

- $\sim 1.4 * 10^{18}$ non-isomorphic complexes,
- $\sim 1.8 * 10^{12}$ non-isomorphic 2-pure complexes.

Needs to be reduced to compute efficiently.

Some reductions #1

Observation

We have $X = Y \cup CA$, where Y has 7 vertices and $A \leq Y$.

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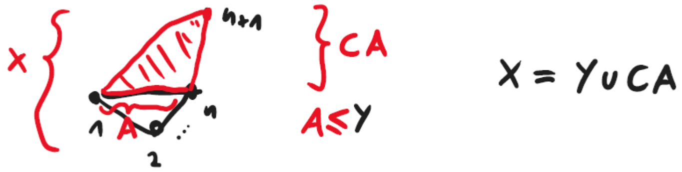

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We can assume X has no free faces.

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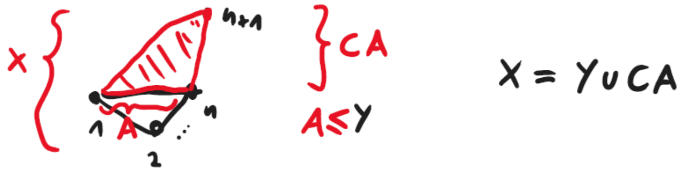


Observation (WLOG #1)

We can assume X has no free faces.

Observation (WLOG #2)

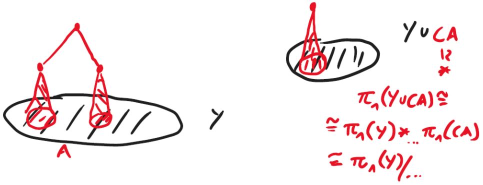
We can assume Y is connected and pure 2-dimensional.

Some reductions #2

Observation

If A has n components and $\pi_1(Y) \cong G$, there is a $H \trianglelefteq G$:

$$\pi_1(X) \cong (G/H) * F_{n-1}.$$



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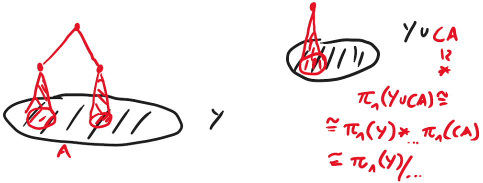

Observation (Almost-WLOG #3)

We can assume $\pi_1(Y) \ncong 1, \mathbb{Z}, \mathbb{Z}_2$.

This loses some information. Only 332710 possible Y remain.

Some reductions #3

Observation (WLOG #4)

We can assume A is at most 1-dimensional.

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Observation (WLOG #5)

We can assume A contains all free edges of Y.

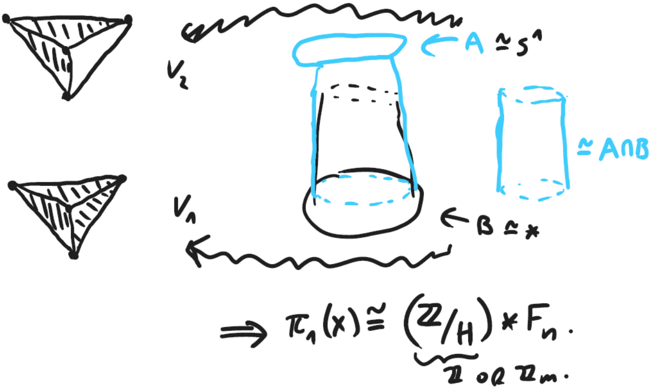
There are (at most) $\sim 4.5 \cdot 10^9$ possible complexes $X = Y \cup CA$.

Testing the complexes

Observation

Suppose the vertices of X can be split as $V = V_1 \cup V_2$, where V_1 span ≥ 3 triangles and V_2 spans ≥ 2 triangles of X. Then:

$$\pi_1(X) \cong \mathbb{Z}_m * F_n, \qquad m \in \mathbb{N}, n \in \mathbb{N}_0.$$



Complexes with ≤ 8 vertices

Procedure (mostly in `Mathematica`):

- Generate the $\sim 4.5 \cdot 10^9$ complexes $X = Y \cup CA$.
- Test for $(4 + 4)$-splitting. This leaves 3807843 complexes.
- Reduce further using "WLOG #3" and cone-and-collapse.
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Result

For any complex X on ≤ 8 vertices, we have

$$\pi_1(X) \cong G * F_n$$

where $n \in \mathbb{N}_0$ and G is one of the following:

- *$G \cong \mathbb{Z}_m$ (for certain m, e.g. $m = 2, 3$),*
- *$G \cong \mathbb{Z} \times \mathbb{Z}$ (π_1 of torus),*
- *$G \cong \mathbb{Z} \rtimes \mathbb{Z}$ (π_1 of Klein bottle),*
- *$G \cong B_3$ (braid group on 3 strands = trefoil knot group).*

Conclusion

Corollary

Any triangulation of the Poincaré sphere has ≥ 13 *vertices.*

(The same conclusion holds for many other manifolds, e.g. T^3.)

Thank you for your attention!