Reeb graph invariants of Morse functions, manifolds and groups

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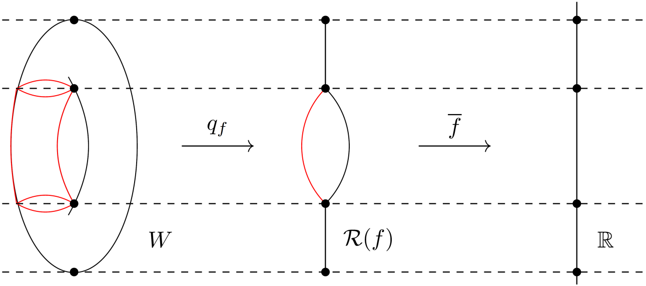
Reeb relation

- W – compact connected smooth manifold of dimension $n \geqslant 2$, M – closed,
- $f\colon W \to [a,b]$ – a smooth function with isolated critical points in Int W and constant on connected components of ∂W (e.g. Morse function)

Definition (G. Reeb 1946 [37], A. Kronrod 1950 [20])

For $x, y \in W$ we say that $x \sim y$ if and only if they are in the same connected component of $f^{-1}(c)$, for some $c \in \mathbb{R}$. The quotient space $\mathcal{R}(f) := W/_{\sim}$ is called the **Reeb graph** of f.



Applications

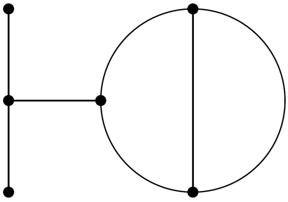
- computer graphics, Y. Shinagawa et al. 1991 [42],
- fundamental role in computational topology (see S. Biasotti et al. 2008 [2]),
- topological data analysis – mapper,
- singularity theory, foliation theory, the study of Morse functions ([1, 10, 40])
- in geometric topology? geometric group theory?

Good orientation of Reeb graphs

A **good orientation** of a graph Γ is the orientation induced by a continuous function $g \colon \Gamma \to \mathbb{R}$ such that

- is strictly monotonic on the edges
- has extrema exactly in the vertices of degree one.

The function $\overline{f} \colon \mathcal{R}(f) \to \mathbb{R}$ such that $f = \overline{f} \circ q_f$, where $q_f \colon W \to \mathcal{R}(f)$ is the quotient map, assigns a good orientation on $\mathcal{R}(f)$.



Graph which not admit a good orientation (V. Sharko [41]).

Realization problems

Problem

For a given manifold W, which graph Γ can be realized as the Reeb graph of a function $f : W \to \mathbb{R}$ with finitely many critical points (or of a Morse function)?

Realization up to:

- isomorphism of oriented graphs – unsolved for $n \geqslant 3$,
- orientation-preserving homeomorphism,
- homotopy equivalence (up to cycle rank).

• The **cycle rank** of a graph Γ = its first Betti number $\beta_1(\Gamma)$.

Proposition (V. Sharko 2006 [41], Y. Masumoto and O. Saeki 2011 [29])

For a graph Γ with good orientation there exist a closed surface Σ and $f : \Sigma \to \mathbb{R}$ with isolated critical points such that $\mathcal{R}(f)$ is isomorphic to Γ.

S-good orientation of graph

Definition

A directed graph has an S-good orientation for $S \subset \mathbb{Z}_{\geqslant 0}$ if it is acyclic and all its sources and sinks have degrees in S.

- A good orientation is a $\{1\}$-good orientation

Theorem (I. Gelbukh 2022 [12])

A connected non-trivial graph Γ admits an S-good orientation if and only if it has no loops and each its leaf block has a non-cut vertex of degree in S.

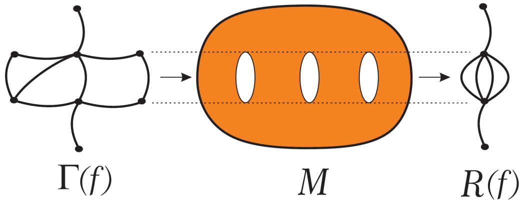
- Thus Γ admits a good orientation if and only if it has no loops and all its leaf blocks are K_2.

- I. Gelbukh showed that Γ is the Reeb graph of a Morse–Bott function if and only if it admits $\{0, 1, 2\}$-good orientation. In fact, any finite graph is homeomorphic to a graph with $\{0, 1, 2\}$-good orientation.

Reeb graph as a subcomplex

Theorem (M. Kaluba, W. Marzantowicz and N. Silva 2015 [19])

There exist a graph $\Gamma(f)$ and an embedding $\iota\colon \Gamma(f) \to W$ such that the composition $q_f \circ \iota\colon \Gamma(f) \to \mathcal{R}(f)$ is a homotopy equivalence, which contracts trees on critical levels to the points.



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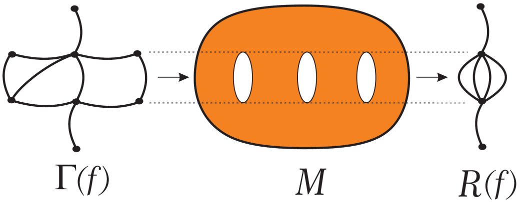

For a connected graph Γ, $\pi_1(\Gamma) \cong F_{\beta_1(\Gamma)}$, where F_r is the free group of rank r.
Thus $(q_f)_\#\colon \pi_1(W) \to \pi_1(\mathcal{R}(f)) \cong F_{\beta_1(\mathcal{R}(f))}$ is surjective and

$$\beta_1(\mathcal{R}(f)) \leqslant \operatorname{corank}(\pi_1(W)) \leqslant \beta_1(W),$$

where $\operatorname{corank}(G)$ is the maximum rank of an epimorphism $G \to F_r$.

Reeb number

Definition

The **Reeb number** $\mathcal{R}(W)$ is the maximum cycle rank among all Reeb graphs of functions on W with finitely many critical points.

In the calculation of $\mathcal{R}(W)$ it suffices to consider simple Morse functions.

• A Morse function is **simple** if each critical value corresponds exactly to a one critical point.

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• Σ_g – orientable, S_g – non-orientable closed surface of genus g.

Lemma (K. Cole-McLaughlin, H. Edelsbrunner et al. 2004 [4])

Let $f : \Sigma \to \mathbb{R}$ be a simple Morse function on a closed surface Σ.

1. *If $\Sigma = \Sigma_g$, then $\beta_1(\mathcal{R}(f)) = g$.*
2. *If $\Sigma = S_g$, then $\beta_1(\mathcal{R}(f)) \leqslant \left\lfloor \frac{g}{2} \right\rfloor$, where $\lfloor x \rfloor$ is the floor of x.*

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Corollary (cf. Kaluba–Marzantowicz–Silva 2015 [19], I. Gelbukh 2018 [10])

$$\mathcal{R}(\Sigma_g) = g \quad \text{and} \quad \mathcal{R}(S_g) = \left\lfloor \frac{g}{2} \right\rfloor.$$

Realization theorems for surfaces

$\Gamma_0 =$ 

If $\mathcal{R}(f)$ is isomorphic to Γ_0, then $f\colon M \to \mathbb{R}$ has only two critical points. Reeb Theorem asserts that M is homeomorphic to the n-dimensional sphere. Conversely, Γ_0 is the Reeb graph of a height function on S^n.

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- Σ – a closed surface, Γ – a finite graph with good orientation

Theorem (Ł. M. 2018 [31])

For $\Gamma \neq \Gamma_0$ there exists a function $f\colon \Sigma \to \mathbb{R}$ with finitely many critical points such that $\mathcal{R}(f) \cong \Gamma$ if and only if $\beta_1(\Gamma) \leqslant \mathcal{R}(\Sigma)$.

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Theorem (Ł. M. 2018 [31])

There exists a Morse function $f\colon \Sigma \to \mathbb{R}$ such that $\mathcal{R}(f) \cong \Gamma$ if and only if

- $g \geqslant \beta_1(\Gamma) + \Delta_2(\Gamma)$, *when Σ is orientable of genus g,*
- $g \geqslant 2\beta_1(\Gamma) + \Delta_2(\Gamma)$, *when Σ is non-orientable of genus g,*

where $\Delta_2(\Gamma)$ is the number of vertices of degree 2 in Γ.

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Theorem (Ł. M. 2023 [33])

There exists a simple Morse function $f\colon \Sigma \to \mathbb{R}$ such that $\mathcal{R}(f) \cong \Gamma$ if and only if $\Delta(\Gamma) \leqslant 3$ and

- *$g = \beta_1(\Gamma)$ and $\Delta_2(\Gamma) = 0$, when Σ is orientable of genus g,*
- *$g = 2\beta_1(\Gamma) + \Delta_2(\Gamma)$, when Σ is non-orientable of genus g,*

where $\Delta(\Gamma)$ is the maximum degree of a vertex in Γ.

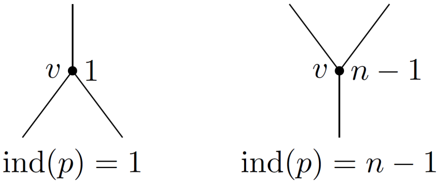
Degree and index correspondence, $n \geqslant 3$

Proposition (G. Reeb 1946 [37])

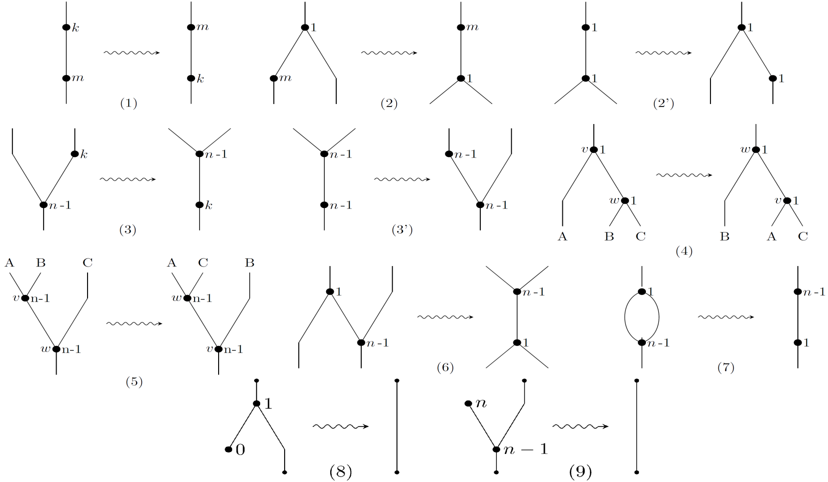
Let $f\colon W \to \mathbb{R}$ be a simple Morse function, p a critical point and $v := q_f(p)$ the vertex in $\mathcal{R}(f)$ which corresponds to p. Then

$$\deg(v) = \begin{cases} 1 & \text{if } \operatorname{ind}(p) = 0 \text{ or } n, \\ 2 \text{ or } 3 & \text{if } \operatorname{ind}(p) = 1 \text{ or } n-1, \\ 2 & \text{in other cases.} \end{cases}$$

$$\operatorname{ind}(p) = \begin{cases} 1 & \text{if } \deg(v) = 3 \text{ and } \deg_{in}(v) = 2, \\ n-1 & \text{if } \deg(v) = 3 \text{ and } \deg_{out}(v) = 2, \end{cases}$$



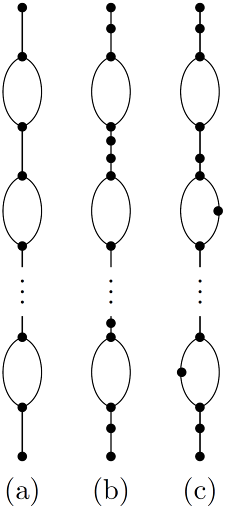
Combinatorial modifications of Reeb graphs



• defined for Reeb graphs of simple Morse functions on *n*-manifolds, $n \geqslant 3$.

• similar operations for orientable surfaces (E. Kudryavtseva 1999 [21] and Fabio–Landi 2016 [8])

Canonical form of graph



- M – a smooth closed manifold of dimension $n \geqslant 3$.
- Any simple Morse function on M can be modified using a finite number of combinatorial modifications to a simple Morse function whose Reeb graph is in a canonical form.
- cf. similar fact for orientable surfaces, E. Kudryavtseva 1999 [21], B. Di Fabio and C. Landi 2016 [8].

Corollary

For any integer $0 \leqslant k \leqslant \mathcal{R}(M)$ there exists a simple Morse function $f : M \to \mathbb{R}$ such that $\beta_1(\mathcal{R}(f)) = k$.

(a) the canonical graph;
(b) graph in a canonical from;
(c) graph not in a canonical form.

Reeb number and corank

Theorem (Ł. M. 2021 [32])

For a closed manifold M the following are equivalent:

1. *There exists a Morse function $g\colon M \to \mathbb{R}$ (simple if M is not an orientable surface) such that $\beta_1(\mathcal{R}(g)) = r$.*
2. *There is an epimorphism $\pi_1(M) \to F_r$.*
3. *There exist disjoint submanifolds $N_1, \ldots, N_r \subset M$ of codimension 1 with product neighbourhoods such that $M \setminus \bigcup_{i=1}^{r} N_i$ is connected.*

The equivalence of (2) and (3) was shown by O. Cornea 1989 [6] and for combinatorial manifolds by W. Jaco 1972 [18]. I. Gelbukh 2018 [10] established the equivalence of (1) and (2) for orientable manifolds.

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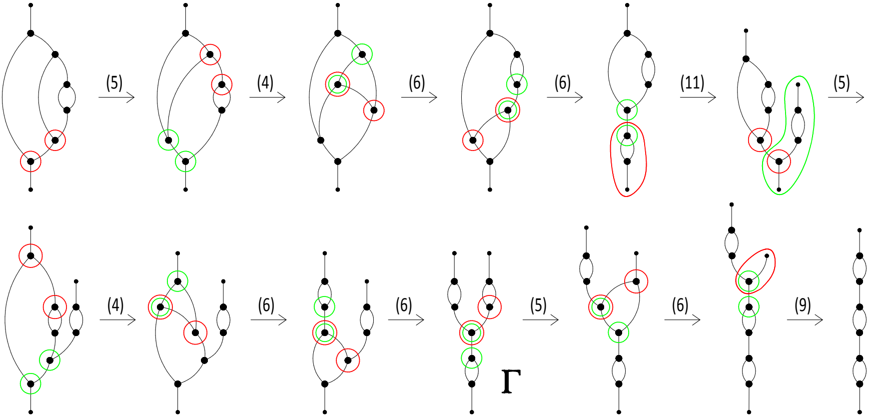
Corollary

$$\mathcal{R}(M) = \operatorname{corank}(\pi_1(M)).$$

Realization theorem

Theorem (Ł. M. 2021 [32])

Let M be a smooth, closed n-manifold, $n \geqslant 3$, and Γ be a finite graph with good orientation such that $\beta_1(\Gamma) \leqslant \mathcal{R}(M)$. Then there exists a Morse function $f\colon M \to \mathbb{R}$ such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ. If $\Delta(\Gamma) \leqslant 3$, then f can be taken to be simple.



Realization up to isomorphism

Theorem (O. Saeki 2022 [38])

Let M be a smooth, closed n-manifold, $n \geqslant 2$, and Γ be a finite oriented graph without loops such that $\beta_1(\Gamma) \leqslant \mathcal{R}(M)$. Then there exists a function $f: M \to \mathbb{R}$ with finitely many critical values such that $\mathcal{R}(f) \cong \Gamma$.

- But f has infinitely many critical points forming n-dimensional submanifolds!

Systems of hypersurfaces

- A **system of hypersurfaces** of size r in W is a sequence $\mathcal{N} = (N_1, \dots, N_r)$ of disjoint, proper ($\partial N_i = N_i \cap \partial W$), framed submanifolds N_i of codimension 1.

- Denote by $P(N_i) \cong N_i \times [-1, 1]$ a product neighbourhood of N_i compatible with the framing and by

$$W|\mathcal{N} := W \setminus \bigcup_{i=1}^{r} \operatorname{Int} P(N_i) \text{ the complement of the system.}$$

Definition (Extended Pontryagin–Thom construction)

A system $\mathcal{N}$ determines a map $f_{\mathcal{N}} \colon W \to \bigvee_{i=1}^{r} \mathrm{S}^1_i$ and induced homomorphism $\varphi_{\mathcal{N}} := (f_{\mathcal{N}})_{\#} \colon \pi_1(W) \to \pi_1(\bigvee_{i=1}^{r} \mathrm{S}^1_i) =: F_r$ as follows:
$f_{\mathcal{N}}$ maps $(x, t) \in P(N_i)$ into $t \in [-1, 1]/_{\{\pm 1\}} = \mathrm{S}^1_i$ and $W|\mathcal{N}$ to the basepoint.

Epimorphisms and independent systems

A system $\mathcal{N} = (N_1, \ldots, N_r)$ of hypersurfaces in W is

- **independent** if $W|\mathcal{N}$ is connected,
- **regular** if each N_i is connected.

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Theorem (W. Marzantowicz and Ł. M. 2020 [28])

Any epimorphism $\varphi\colon \pi_1(W) \to F_r$ is induced by a regular and independent system of hypersurfaces.

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• **Reeb epimorphism** $(q_f)_\#\colon \pi_1(W) \to \pi_1(\mathcal{R}(f)) \cong F_r$ of a Morse function f – induced by the quotient map $q_f\colon W \to \mathcal{R}(f)$.

• Connected components of level sets corresponding to points on edges outside some spanning tree of $\mathcal{R}(f)$ form a regular and independent system $\mathcal{N}$ such that $\varphi_{\mathcal{N}} = (p_T)_\# \circ (q_f)_\#$, where p_T contracts T.

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Problem

Which epimorphism $\pi_1(W) \to F_r$ can be represented as the Reeb epimorphism?

Representation of epimorphism

Theorem (W. Marzantowicz and Ł. M. 2020 [28])

For an epimorphism $\varphi\colon \pi_1(W) \to F_r$ *the following are equivalent:*

- $\varphi = \varphi_{\mathcal{N}}$ *for an independent and regular system* $\mathcal{N}$ *without boundary;*
- φ *is factorized through* $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$*, where* $\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$ *is the normal closure in* $\pi_1(W)$ *of loops from* ∂W*;*
- φ *is represented as the Reeb epimorphism of a Morse function on* W

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- φ *is represented as the Reeb epimorphism of a Morse function on W*

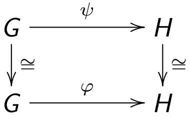
Corollary

$$\mathcal{R}(W) = \operatorname{corank}(\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)})$$

is equal to the maximum size of an independent and regular system of hypersurfaces without boundary in W.

Moreover, $\operatorname{corank}(\pi_1(W))$ is the maximum size of an independent and regular system in W (W. Jaco 1972 [18]).

(Strong) equivalence of epimorphisms



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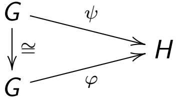

strong equivalence $\simeq$

(Strong) equivalence of epimorphisms

$$\begin{array}{ccc} G & \xrightarrow{\psi} & H \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ G & \xrightarrow{\varphi} & H \end{array}$$

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strong equivalence $\simeq$

Theorem (Stallings–Jaco–Waldhausen–Hempel, [16, 17])

The Poincaré conjecture holds if and only if for each $g \geqslant 2$ any two epimorphisms $\pi_1(\Sigma_g) \to F_g \times F_g$ are equivalent.

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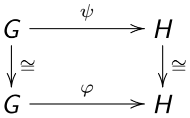

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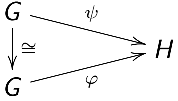

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Theorem (R. Grigorchuk, P. Kurchanov and H. Zieschang 1992 [13, 14, 15])

If Σ is a closed surface, then the numbers p and q of p and q of equivalence and strong equivalence, respectively, of epimorphisms $\pi_1(\Sigma) \to F_r$ are finite. More precisely, if $\Sigma = S_{2g}$, then

- $p = 2$ *and* $q = 2^r$ *if* $r < g = \operatorname{corank}(\pi_1(S_{2g}))$,
- $p = 1$ *and* $q = 2^r - 1$ *if* $r = g$,

and $p = q = 1$ *in other cases.*

Framed cobordism of systems of hypersurfaces

- Two systems $\mathcal{N} = (N_1, \ldots, N_r)$ and $\mathcal{N}' = (N_1', \ldots, N_r')$ of hypersurfaces in a closed manifold M of the same size r are **framed cobordant** (as systems of hypersurfaces) if there are r **disjoint** framed cobordisms $W_i \subset M \times [0, 1]$ between N_i and N_i'. Equivalently, $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.

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- $\mathcal{H}_r^{fr}(M)$ — the set of framed cobordism classes of all independent and regular systems of hypersurfaces in M of size r.

- The natural map

$$\begin{array}{rcl} \mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)} & \to & \mathrm{Epi}(\pi_1(M), F_r)/_{\simeq} \\ [\mathcal{N}] & \mapsto & [\varphi_{\mathcal{N}}] \end{array}$$

is surjective, and it is bijection for surfaces or hyperbolic manifolds.

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is surjective, and it is bijection for surfaces or hyperbolic manifolds.

$$\mathcal{H}_r^{fr}(S_{2m})/_{\mathrm{Diff}_\bullet(S_{2m})} = \begin{cases} \{[\mathcal{N}_0], [\mathcal{N}_J] \,:\, \varnothing \neq J \subset \{1, \ldots, r\}\} & \text{for } r < m, \\ \{[\mathcal{N}_J] \,:\, \varnothing \neq J \subset \{1, \ldots, r\}\} & \text{for } r = m, \end{cases}$$

where $S_{2m}|\mathcal{N}_J$ is orientable, but $(S_{2m}|\mathcal{N}_J) \cup P(N_j)$ is non-orientable only for $j \in J$, and $S_{2m}|\mathcal{N}_0$ is non-orientable for $r < m$.

Reeb epimorphisms

Theorem (W. Marzantowicz and Ł. M. 2020 [28]; cf. O. Saeki 2022 [38])

Let Γ be a finite connected graph with good orientation and $\Delta_1(\Gamma) \geqslant |\pi_0(\partial W)|$, and let $\varphi\colon \pi_1(W) \to \pi_1(\Gamma)$ be an epimorphism. Then there is a Morse function $f: M \to \mathbb{R}$ such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ and under this identification the Reeb epimorphism of f is equal to φ.
Moreover, if W is not a surface and $\Delta(\Gamma) \leqslant 3$, then f can be taken to be simple.

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Moreover, if W is not a surface and $\Delta(\Gamma) \leqslant 3$, then f can be taken to be simple.

Theorem (W. Marzantowicz and Ł. M. 2020 [28])

Let Σ be a closed surface, $\Delta(\Gamma) \leqslant 3$ and let $\varphi\colon \pi_1(\Sigma) \to F_r$ be an epimorphism. Then φ is the Reeb epimorphism of a simple Morse function on Σ if and only if

- *$\beta_1(\Gamma) = g$, when Σ is orientable of genus g.*
- *$1 = 1$ (no requirements), when Σ is non-orientable of odd genus.*
- *$\beta_1(\Gamma) = g$, or $\beta_1(\Gamma) < g$ and ψ belongs to a unique strong equivalence class of epimorphisms represented by systems of hypersurfaces whose complement is non-orientable, when Σ is non-orientable of even genus $2g$.*

Topological conjugation of Morse functions

• Functions $f_1, f_2 \colon M \to \mathbb{R}$ are **topologically conjugate** if there exist a self-homeomorphism $h \colon M \to M$ and an orientation-preserving homeomorphism $\eta \colon \mathbb{R} \to \mathbb{R}$ such that $f_1 = \eta \circ f_2 \circ h$.

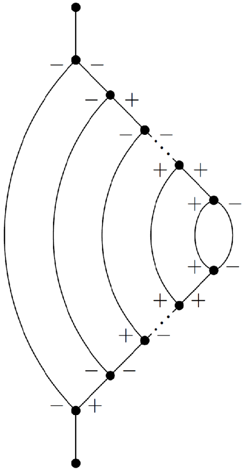
• The map h induces a homeomorphism $\overline{h} \colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$, which is in fact an isomorphism of oriented graphs if f_i are simple Morse functions.

Theorem (E. Kulinich 1998 [22], V. Sharko 2003 [40])

Let Σ_g be a closed orientable surface of genus g. Two simple Morse functions are topologically conjugate by $h \colon \Sigma_g \to \Sigma_g$ if and only if their Reeb graphs are isomorphic through $\overline{h}$.

Non-orientable surfaces

• $f: S_g \to \mathbb{R}$ – a simple Morse function on non-orientable surface of genus g,
• equip $\mathcal{R}(f)$ with two signs $+$ / $-$ at each vertex of degree 3 near two incoming or outgoing edges, which correspond to the way of attaching the 1-handle.



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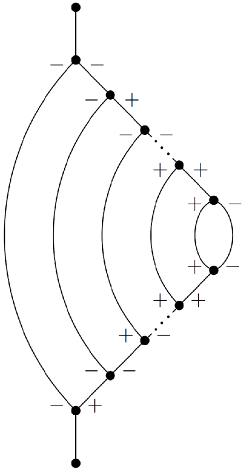

Theorem (D. Lychak and A. Prishlyak 2009 [23])

Two simple Morse functions on a closed non-orientable surface are topologically conjugate if and only if their Reeb graphs are isomorphic and it is possible to obtain identical signs on their equipped Reeb graphs by uses of the following operation:

- *choose an edge and reverse signs on its ends.*

Strong-equivalence for non-orientable surfaces

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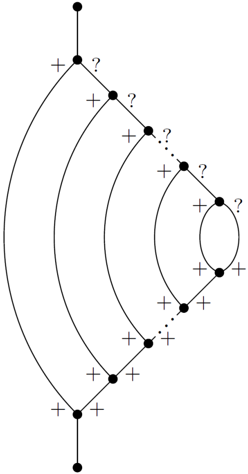

• There is 2^r equivalence classes of graphs with signs in the initial form with r cycles,

• the case with only pluses corresponds to an orientable surface.

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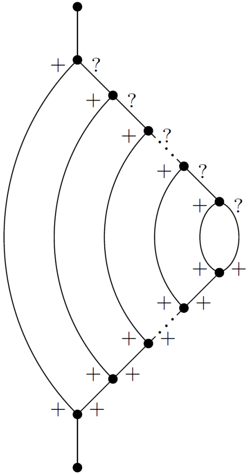

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Topological conjugation, framed cobordism and strong equivalence

- Fix a graph Γ with good orientation and points $a_1, \dots a_r$, $r = \beta_1(\Gamma)$, on different edges outside a spanning tree $T \subset \Gamma$.

- $\mathcal{M}(M, \Gamma)$ – the set of simple Morse functions $f\colon M \to \mathbb{R}$ such that $\mathcal{R}(f) \cong \Gamma$.

$$\begin{array}{ccccc} \mathcal{M}(M,\Gamma)/_{\mathrm{t.c.}} & \to & \mathcal{H}^{fr}_r(M)/_{\mathrm{Diff}_\bullet(M)} & \to & \mathrm{Epi}(\pi_1(M), F_r)/_{\simeq} \\ [f\colon M \to \mathbb{R}] & \mapsto & [\mathcal{N} = (N_1, \dots, N_r)] & \mapsto & \varphi_{\mathcal{N}} \\ & & N_i = q_f^{-1}(a_i) & & \end{array}$$

Corank of a finitely generated group

- corank(G) – the maximum r for which there exists an epimorphism $G \to F_r$.

$$\operatorname{corank}(G) \leqslant \operatorname{rank}_{\mathbb{Z}} \operatorname{Ab}(G)$$

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$$\operatorname{corank}(G * H) = \operatorname{corank}(G) + \operatorname{corank}(H),$$
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- For the short exact sequence $1 \to N \to G \to H \to 1$ of finitely generated groups:

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Problem

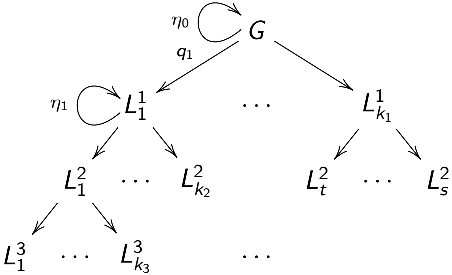
Find an algorithmic method of calculating the corank.

• J. Stallings 1992 [43] proposed to use systems of hypersurfaces – we are not aware of such method.

Algebraic description of $\mathrm{Hom}(G, F_r)$

There is a description of the structure of the set $\mathrm{Hom}(G, F_r)$ in terms of **Makanin–Razborov diagrams** [26, 36, 3, 39]. They come from the studies of sets of solutions to equations defined over a free group.

All homomorphisms $G \to F_r$ are encoded into a finite diagram of groups, where each edge represents a proper quotient map and groups at the ends of branches are free.



Every $\varphi \in \mathrm{Hom}(G, F_r)$ is **M–R factorized** through some branch of the diagram, i.e. it can be written as the composition of quotient maps q_i, modular automorphisms $\eta_i \in \mathrm{Mod}(L_i)$ and some $L_k \to F_r$, for L_k a free group.

Reeb graphs with corank cycles

Problem

How to find/construct a function $f : M \to \mathbb{R}$ such that $\beta_1(\mathcal{R}(f)) = \mathrm{corank}(\pi_1(M))$? What are necessary conditions for such f?

- For brevity, let's call such a function **maximal**.

- k_i – the number of critical points of index i of a simple Morse function f.

- If f has exactly two extrema, then the number $\Delta_2(\mathcal{R}(f))$ of vertices of degree 2 in $\mathcal{R}(f)$ is equal to

$$\Delta_2(\mathcal{R}(f)) = k_1 + \ldots + k_{n-1} - 2\beta_1(\mathcal{R}(f))$$

Degree 2 vertices

Proposition

If $f : M \to \mathbb{R}$ is a simple Morse function on a closed n-manifold M, then

$$\Delta_2(\mathcal{R}(f)) \cong \chi(M) \mod 2,$$

$$\Delta_2(\mathcal{R}(f)) \geqslant 2(\mathrm{rank}(\pi_1(M)) - \mathrm{corank}(\pi_1(M))), \tag{1}$$

$$\Delta_2(\mathcal{R}(f)) \geqslant \sum_{i=1}^{n-1} \mathrm{rank}_R H_i(M, R) - 2\,\mathrm{corank}(\pi_1(M)), \tag{2}$$

$$\Delta_2(\mathcal{R}(f)) \geqslant \mathrm{cat}(M) - 2\,\mathrm{corank}(\pi_1(M)) - 2. \tag{3}$$

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Definition

By $\Delta_2(M)$ we denote the minimal number of vertices of degree 2 in Reeb graphs of simple Morse functions on a manifold M.

- $\Delta_2(\Sigma_g) = 0$, $\Delta_2(S_{2k}) = 0$ and $\Delta_2(S_{2k+1}) = 1$.

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Example ($M = T^n$ – the n-dimensional torus)

1. $\Delta_2(T^n) \geqslant 2(n-1)$,
2. $\boxed{\Delta_2(T^n) \geqslant 2^n - 4,}$
3. $\Delta_2(T^n) \geqslant (n+1) - 4 = n - 3.$

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Example ($M = L_p \# L_q$, the connected sum of 3-dim. lens spaces, $(p, q) = 1$)

1. $\Delta_2(L_p \# L_q) \geqslant 2 \cdot (2 - 0) = 4,$
2. $2 \geqslant \operatorname{rank}_R H_1(M, R) + \operatorname{rank}_R H_2(M, R),$
3. $\Delta_2(L_p \# L_q) \geqslant 4 - 2 = 2.$

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Example ($M = \Sigma_g \times S^{n-2}$, $n \geqslant 3$)

1. $\Delta_2(\Sigma_g \times S^{n-2}) \geqslant 2(2g - g) = 2g$,
2. $\boxed{\Delta_2(\Sigma_g \times S^{n-2}) \geqslant 4g + 2 - 2g = 2g + 2,}$
3. $3 + 2 - 2g - 2 \geqslant \mathrm{cat}(\Sigma_g \times S^{n-2}) - 2g - 2$.

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Example ($\mathbb{R}P^n$ and Σ – the Poincaré homology sphere)

1. $\Delta_2(\mathbb{R}P^n) \geqslant 2,$
2. $\boxed{\Delta_2(\mathbb{R}P^n) \geqslant n - 1,}$
3. $\boxed{\Delta_2(\mathbb{R}P^n) \geqslant (n + 1) - 2.}$

1. $\boxed{\Delta_2(\Sigma) \geqslant 2,}$
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Problem

Can (3) be better than (2) and (1) or not? In particular, does the following inequality hold?

$$\max\{\, 2\operatorname{rank}(\pi_1(M)), \sum_{i=1}^{n-1} \operatorname{rank}_R H_i(M, R) \,\} \geqslant \operatorname{cat}(M) - 2$$

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Example ($\Sigma \neq S^n$ – a homology sphere)

1. $\Delta_2(\Sigma) \geqslant \mathrm{rank}(\pi_1(\Sigma)) \geqslant 2,$
2. $\Delta_2(\Sigma) \geqslant 0,$
3. $\Delta_2(\Sigma) \geqslant \mathrm{cat}(\Sigma) - 2 \geqslant 4 - 2 = 2.$

Reeb graphs of orientable 3-manifolds

Moreover, if M is an orientable 3-manifold with Heegaard genus $g(M)$, then

$$2g(M) \geqslant \Delta_2(\mathcal{R}(f)) \geqslant 2(g(M) - \operatorname{corank}(\pi_1(M))). \tag{4}$$

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Corollary

The following are equivalent:

1. *There is a maximal simple Morse function on M with minimum number of critical points.*
2. $\Delta_2(M) = 2(g(M) - \operatorname{corank}(\pi_1(M)))$.
3. *Any simply Morse function $f : M \to \mathbb{R}$ with exactly two extrema and $\Delta_2(\mathcal{R}(f)) = \Delta_2(M)$ is maximal and has the minimum number of critical points.*

Reeb graphs of orientable 3-manifolds

Theorem (Ł. M. 2023 [33])

For orientable 3-manifolds

- $\Delta_2(M) = 0$ *if and only if* $M = \#^g\, S^2 \times S^1$.
- $\Delta_2(M) = 2$ *if and only if* $M = (\#^g\, S^2 \times S^1)\# L_p$, *where* L_p *is a lens space.*

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Problem

Is it true that

$$\Delta_2(M \# N) = \Delta_2(M) + \Delta_2(N)?$$

Group presentation invariant

- $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ – a group presentation with $\text{rank}(\mathcal{P}) = n$ generators and m relators $r_i = r_i(x_1, \ldots, x_n)$. It has deficiency $\text{def}(P) = n - m$.
- For a finitely presented group G, $\text{def}(G) = \max(\text{def}(\mathcal{P}))$ over all $\mathcal{P} \cong G$.

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Theorem (D. Epstein 1961 [7])

If M is a closed orientable 3-manifold, then $\mathrm{def}(\pi_1(M)) = 0$

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Define $\Omega = \Omega(\mathcal{P})$ to be the minimum positive number such that for $1 \leqslant i \leqslant \min\{n - \Omega, m\}$ the relator r_i can be written as a word in only first $\Omega + i - 1$ generators, i.e. $r_i = r_i(x_1, \ldots, x_{\Omega+i-1})$.
For $n \geqslant \text{rank}(G)$,

$$\Omega_n(G) := \min\left\{\Omega(\mathcal{P}) : \; G \cong \mathcal{P}, \text{def}(\mathcal{P}) = \text{def}(G) \text{ and } \text{rank}(\mathcal{P}) = n\right\}.$$

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Example ($G = \pi_1(\Sigma)$ for the Poincaré sphere Σ)

- $\langle a, b, c \mid a^2 = b^3 = c^5 = abc\rangle$ write as $\mathcal{P}_1 = \langle a, b, c \mid a^{-1}bc, b^{-3}abc, abc^{-4}\rangle$, so $\Omega(\mathcal{P}_1) = 3$.

Group presentation invariant

Definition

Define $\Omega = \Omega(\mathcal{P})$ to be the minimum positive number such that for $1 \leqslant i \leqslant \min\{n - \Omega, m\}$ the relator r_i can be written as a word in only first $\Omega + i - 1$ generators, i.e. $r_i = r_i(x_1, \ldots, x_{\Omega+i-1})$.
For $n \geqslant \operatorname{rank}(G)$,

$$\Omega_n(G) := \min \{\Omega(\mathcal{P}) : \; G \cong \mathcal{P}, \operatorname{def}(\mathcal{P}) = \operatorname{def}(G) \text{ and } \operatorname{rank}(\mathcal{P}) = n\}.$$

and

$$\Omega(G) := \min_n \Omega_n(G).$$

Example ($G = \pi_1(\Sigma)$ for the Poincaré sphere Σ)

- $\langle a, b, c \mid a^2 = b^3 = c^5 = abc\rangle$ write as $\mathcal{P}_1 = \langle a, b, c \mid a^{-1}bc, b^{-3}abc, abc^{-4}\rangle$, so $\Omega(\mathcal{P}_1) = 3$.
- $\langle b, c \mid (bc)^2 = b^3 = c^5\rangle$ write as $\mathcal{P}_2 = \langle b, c \mid b^{-2}cbc, bcbc^{-4}\rangle$, so $\Omega(\mathcal{P}_2) = 2$.

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- $\mathcal{P}_3 = \langle b, c, d \mid d, b^{-2}cbc, bcbc^{-4}\rangle$, so $\Omega(\mathcal{P}_3) = 2$.

Group presentation invariant

Theorem (Ł. M. 2023 [33])

Let $f : M \to \mathbb{R}$ be a simple Morse function on an orientable closed 3-manifold M with k_i critical points of index i and $k_0 = 1$. Then

$$\Omega_{k_1}(\pi_1(M)) \leqslant k_1 - \beta_1(\mathcal{R}(f))$$

and

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Lemma

If G is a non-trivial, non-free, torsion-free group, then $\Omega(G) \geqslant 2$.

Corollary

If $\pi_1(M)$ is torsion-free and $g(M) = \operatorname{corank}(\pi_1(M)) + 1 \geqslant 2$, then $\Delta_2(M) \geqslant 4$, and so there is no maximal Morse function with minimum number of critical points.

- For example, the Heisenberg manifold $M = H(3,\mathbb{R})/H(3,\mathbb{Z})$ has $\Delta_2(M) = 4$.

S^1-bundles over a surface

- Let M_e be a circle bundle over Σ_g with Euler number $e \in \mathbb{Z}$, e.g. $M_0 = \mathrm{S}^1 \times \Sigma_g$. Then $\mathrm{corank}(\pi_1(M_e)) = g$.

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- Moreover, $g(M_e) = 2g + 1$ for $e \neq \pm 1$ and

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- If $e = \pm 1$, then $g(M_{\pm 1}) = 2g$ and for $h = \Pi_i [a_i, b_i]$

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• A word $x_{i_1} \ldots x_{i_k}$ in a free group $F_n = \langle x_1, \ldots, x_n \,|\, \rangle$ is **cyclically reduced** if it is reduced and $x_{i_k} x_{i_1} \neq 1$.

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Theorem (*Freiheitssatz*, W. Magnus 1930 [24, 25])

If r is a cyclically reduced word in F_n that contains x_i, then every non-trivial element of the normal closure of r also contains x_i.

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Sketch of the proof.

For $h = \Pi_i [a_i, b_i]$

$$\pi_1(M_{\pm 1}) = \langle a_1, b_1, \ldots, a_g, b_g \,|\, [a_i, h^e] = [b_i, h^e] = 1 \rangle,$$

so we have the quotient map $\varphi \colon \pi_1(M_{\pm 1}) \to \pi_1(\Sigma_g)$ such that $\varphi(h) = 1$.

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Fixing generators $a_1, b_1, \ldots, a_g, b_g$ of F_{2g} and the canonical quotient $\psi\colon F_{2g} \to \pi_1(M_{\pm 1})$, the normal closure of h in F_{2g} is equal to $\ker(\varphi \circ \psi)$. So any relator of $\pi_1(M_{\pm 1})$ in F_{2g} is also contained in the normal closure of h. By *Freiheitssatz* each such relator needs all generators.

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In general, one needs to prove that all generating sets of $\pi_1(M_{\pm 1})$ of rank $2g$ are Nielsen equivalent.

□

S^1-bundles over a surface

Combining

$$\Omega_{k_1}(\pi_1(M)) \leqslant k_1 - \beta_1(\mathcal{R}(f))$$

and

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Corollary

Any simple Morse function $f : M_{\pm 1} \to \mathbb{R}$ with the minimum number of critical points has no cycles in its Reeb graph, i.e. $\beta_1(\mathcal{R}(f)) = 0$, and $\Delta_2(\mathcal{R}(f)) = 4g$.

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$$\Delta_2(\mathcal{R}(f')) = \Delta_2(M_{\pm 1}) = 2g + 2.$$

C^2-topology

Theorem (Ł. M.)

For any $f \in C^\infty(M, \mathbb{R})$ there is a number $\delta > 0$ such that $\beta_1(\mathcal{R}(f)) \leqslant \beta_1(\mathcal{R}(g))$ for all functions g which are δ-close to f in C^2-topology. In particular, the subspace of maximal smooth functions $f \in C^\infty(M, \mathbb{R})$ is open.

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Corollary

There is an embedding $M \subset \mathbb{R}^N$ such that the set of points p, for which $L_p \colon M \to \mathbb{R}$ given by $L_p(x) = ||x - p||^2$ is maximal, has a positive measure.

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Problem

Is it true for any embedding $M \subset \mathbb{R}^n$? How to define a subset P of $\mathbb{R}^N$ such that L_p is maximal for almost all $p \in P$?

Thank you!

References I

[1] E. B. Batista, J. C. F. Costa and J. J. Nuño-Ballesteros, *The Reeb graph of a map germ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with isolated zeros*, Proc. Edinb. Math. Soc. 60 (2017), no. 2, 319–348.

[2] S. Biasotti, D. Giorgi, M. Spagnuolo and B. Falcidieno, *Reeb graphs for shape analysis and applications*, Theoret. Comput. Sci. 392 (2008), 5–22.

[3] M. Bestvina and M. Feighn, *Notes on Sela's work: Limit groups and Makanin–Razborov diagrams*, Geometric and Cohomological Methods in Group Theory, London Math. Soc. Lecture Note Ser. 358 (2009), 1–29.

[4] K. Cole-McLaughlin, H. Edelsbrunner, J. Harer, V. Natarajan and V. Pascucci, *Loops in Reeb graphs of 2-manifolds*, Discrete Comput. Geom. 32 (2004), 231–244.

[5] D. J. Collins and H. Zieschang, *Combinatorial Group Theory and Fundamental Groups*, Encyclopaedia of Mathematical Sciences 58, Algebra VII, Springer 1993.

[6] O. Cornea, *The genus and the fundamental group of high dimensional manifolds*, Stud. Cerc. Mat. 41 (1989), no. 3, 169–178.

[7] D. B. A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford 12 (1961), 205–212.

References II

[8] B. Di Fabio and C. Landi, *The edit distance for Reeb graphs of surfaces*, Discrete Comput. Geom. 55 (2016), 423–461.

[9] I. Gelbukh, *The co-rank of the fundamental group: The direct product, the first Betti number, and the topology of foliations*, Math. Slovaca 67 (2017), 645–656.

[10] I. Gelbukh, *Loops in Reeb graphs of n-manifolds*, Discrete Comput. Geom. 59 (2018), no. 4, 843–863.

[11] I. Gelbukh, *Approximation of Metric Spaces by Reeb Graphs: Cycle Rank of a Reeb Graph, the Co-rank of the Fundamental Group, and Large Components of Level Sets on Riemannian Manifolds*, Filomat 33 (2019), no. 7, 2031–2049.

[12] I. Gelbukh, *Criterion for a graph to admit a good orientation in terms of leaf blocks*, Monatsh. Math. 198 (2022), no. 1, 61–77.

[13] R. I. Grigorchuk and P. F. Kurchanov, *Classification of epimorphisms of fundamental groups of surfaces onto free groups*, Mat. Zametki 48 (1990), no. 2, 26–35.

[14] R. I. Grigorchuk and P. F. Kurchanov, *Some questions of group theory related to geometry*, Encyclopaedia of Mathematical Sciences 58, Algebra VII, Spinger 1993.

References III

[15] R. I. Grigorchuk, P. F. Kurchanov and H. Zieschang, *Equivalence of homomorphisms of surface groups to free groups and some properties of 3-dimensional handlebodies*, Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989), 521–530, Contemp. Math. 131, Part 1, Amer. Math. Soc., Providence, RI, 1992.

[16] J. Hempel, 3-*manifolds*, Ann. of Math. Studies, no. 86, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1976.

[17] W. Jaco, *Heegaard splittings and splittings homomorphisms*, Trans. Amer. Math. Soc. 144 (1969), 365–379.

[18] W. Jaco, *Geometric realizations for free quotients*, J. Aust. Math. Soc. 14 (1972), no. 4, 411–418.

[19] M. Kaluba, W. Marzantowicz and N. Silva, *On representation of the Reeb graph as a sub-complex of manifold*, Topol. Methods Nonlinear Anal. 45 (2015), no. 1, 287–307.

[20] A. S. Kronrod, *On functions of two variables*, Uspekhi Mat. Nauk 5 (1950), 24–134.

[21] E. A. Kudryavtseva, *Reduction of Morse functions on surfaces to canonical form by smooth deformation*, Regul. Chaotic Dyn. 4 (1999), no. 3, 53–60.

References IV

[22] E. V. Kulinich, *On topologically equivalent Morse functions on surfaces*, Methods Funct. Anal. Topology 4 (1998), no. 1, 59–64.

[23] D. P. Lychak and A. O. Prishlyak, *Morse functions and flows on nonorientable surfaces*, Methods Funct. Anal. Topology 15 (2009), no. 3, 251–258.

[24] W. Magnus, *Über diskontinuierliche Gruppen mit einer definierenden Relation (Der Freiheitssatz)*, J. Reine Angew. Math. 163, 141–165 (1930).

[25] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Dover Publications, 1976.

[26] G. S. Makanin, *Equations in a free group*, Math. USSR Izvestiya 21 (1983), no. 3, 483–546.

[27] J. Martinez-Alfaro, I. S. Meza-Sarmiento and R. Oliveira, *Topological classification of simple Morse Bott functions on surfaces*, Contemp. Math. AMS 675 (2016), 165–179.

[28] W. Marzantowicz and Ł. P. Michalak, *Relations between Reeb graphs, systems of hypersurfaces and epimorphisms onto free groups*, preprint (2020), arXiv:2002.02388.

[29] Y. Masumoto and O. Saeki , *A smooth function on a manifold with given Reeb graph*, Kyushu J. Math. 65 (2011), No. 1, 75–84.

[30] F. Memoli and O. B. Okutan, *The distortion of the Reeb quotient map on Riemannian manifolds*, preprint (2018), arXiv:1801.01562.

[31] Ł. P. Michalak, *Realization of a graph as the Reeb graph of a Morse function on a manifold*, Topol. Methods Nonlinear Anal. 52 (2018), no. 2, 749–762.

[32] Ł. P. Michalak, *Combinatorial modifications of Reeb graphs and the realization problem*, Discrete Comput. Geom. 65 (2021), 1038–1060.

[33] Ł. P. Michalak, *Reeb graph invariants of Morse functions and 3-manifold groups*, preprint (2023), https://www.researchgate.net/publication/367568801_Reeb_graph_invariants_of_Morse_functions_and_3-manifold_groups.

[34] J. W. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.

[35] J. W. Milnor, *Topology from the differentiable viewpoint*, The University Press of Virginia, 1965.

[36] A. Razborov, *On systems of equations in a free group*, Math. USSR Izvestiya 25 (1985), 115–162.

References VI

[37] G. Reeb, *Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique*, C. R. Acad. Sci. Paris 222 (1946), 847–849.

[38] O. Saeki, *Reeb spaces of smooth functions on manifolds*, Int. Math. Res. Not. 2022 (2022), no. 11, 8740–8768.

[39] Z. Sela, *Diophantine geometry over groups I: Makanin–Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. 93 (2001), 31–105.

[40] V. V. Sharko, *Smooth and topological equivalence of functions on surfaces*, Ukrainian Math. J. 55 (2003), no. 5, 832–846.

[41] V. V. Sharko, *About Kronrod–Reeb graph of a function on a manifold*, Methods Funct. Anal. Topology 12 (2006), 389–396.

[42] Y. Shinagawa, T. L. Kunii and Y. L. Kergosien, *Surface coding based on Morse theory*, IEEE Comput. Graphics Appl. 11 (1991), 66–78.

[43] J. R. Stallings, *Problems about Free Quotients of Groups*, Geometric group theory, Proceedings of a special research quarter at the Ohio State University, spring 1992, 165–182, de Gruyter, 1995.