Minimal triangulation of finite group actions

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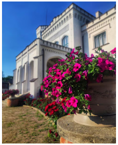
from to Banach Center INSTITUTE OF MATHEMATICS PAS at 

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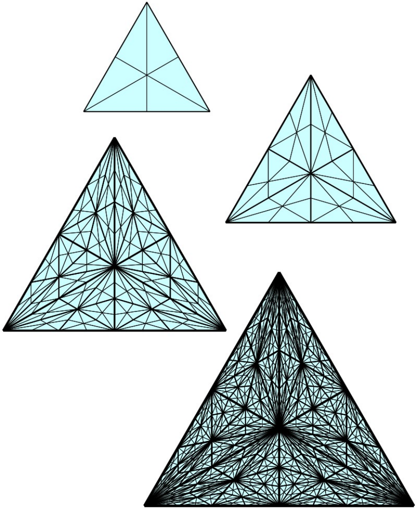
Abstract

Let G be a finite group acting on a closed manifold M. We estimate the size of a minimal triangulation of M for which the action of G is simplicial and regular. We show that the number of vertices of such triangulations are bounded below by the *G-covering type* of M, which is defined as the minimal cardinality of a G-equivariant good cover of a space that is G-homotopy equivalent to M.
The G-covering type is a G-homotopy invariant, so it can be estimated by other G-invariants like the equivariant LS-category, G-genus and the multiplicative structure of any equivariant cohomology theory. In particular, we give a complete description of the number of vertices and their orbits for orientation preserving actions on orientable surfaces.

Definition ((1) see [Bredon, Sec. II.1])

1. A *simplicial G-complex* is a simplicial complex K together with an action of G on K by simplicial maps.
2. A simplicial G-complex K is *regular* if the action of G on K satisfies the following conditions:
   - R1) If vertices v and gv belong to the same simplex in K, then $v = gv$.
   - R2) If $\langle v_0, \ldots, v_n \rangle$ is a simplex of K and if for some choice of $g_0, \ldots, g_n \in H \leq G$ the points $g_0 v_0, \ldots, g_n v_n$ also span a simplex of K, then there exist $g \in H$, such that $gv_i = g_i v_i$, for $i = 0, \ldots n$ (in other words, $\langle g_0 v_0, \ldots, g_n v_n \rangle = g\langle v_0, \ldots, v_n \rangle$).

The regularity condition is quite stringent. For example, neither R1 nor R2 hold for the $\mathbb{Z}_3$-action that rotates the 2-simplex. Furthermore, R1 is satisfied for the induced action on the barycentric subdivision of the 2-simplex, but R2 is not.



Proposition ((2) [Bredon, Prop. II.1.1])

If K is any simplicial G-complex, then the induced action on the barycentric subdivision K' satisfies condition R1. Moreover, if the action of G on K satisfies R1, then the induced action on K' satisfies R2. Therefore, any simplicial action of G on K induces a regular action on the second barycentric subdivision of K.

By condition R2, if two n-simplices in K have vertices from the same set of orbits, then they belong to an orbit of the action of G on K. Thus, if K is a regular G-complex, then one can naturally build a quotient simplicial complex K/G whose vertices are the orbits of the action of G on the vertices of K, and whose simplices are the orbits of the action of G on the simplices of K. Clearly, the geometric realization $|K/G|$ of the quotient complex is homeomorphic to the quotient space $|K|/G$.

Yang [Yang] has introduced an analogous notion for G-covers.

Definition ((3))

An open G-cover $\mathcal{U}$ of a G-space X is *regular* if the following conditions hold:

RC1) For every $U \in \mathcal{U}$ and $g \in G$, either $U = gU$ or $U \cap gU = \emptyset$

RC2) If $U_0, \ldots, U_n$ are elements of $\mathcal{U}$ with non-empty intersection and if for some choice of elements $g_0, \ldots, g_n \in H \leq G$ the intersection of sets $g_0 U_0, \ldots, g_n U_n$ is also non-empty, then there exists $g \in H$ such that $gU_i = g_i U_i$ for $i \leq n$.

In short, $\mathcal{U}$ is a regular G-cover if its nerve $N(\mathcal{U})$ is a regular G-complex.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open G-cover of G-space X. For any subgroup $H \subset G$ and $\alpha \in I$, let $U_\alpha^H = U_\alpha \cap X^H$. Denote by $\mathcal{U}^H$ the collection of $\{U_\alpha^H\}_{\alpha \in I}$. It is clear that $\mathcal{U}^H$ is an open cover of X^H. After [Yang], we define.

Definition ((4) Equivariant good cover I)

An G-cover $\mathcal{U}$ is called an G-equivariant good cover, or shortly a good G-cover, of X if it is a regular G-cover (see Definition 3) and $\mathcal{U}^H$ is a good cover of X^H for all subgroups $H \subset G$.

Theorem 2.11 of [Yang]: every smooth G-manifold has a good G-cover.
Another natural extension onto the equivariant case.

Definition ((5) Equivariant good cover II)

A regular open G-cover $\mathcal{U}$ split into orbits $\tilde{U} = GU$ is said to be a good G-cover if all orbits $\tilde{U}$ of elements of $\mathcal{U}$ and all their non-empty finite intersections are G-contractible.

Remark (6)

Directly from the definition of G-good cover $\mathcal{U}$ (Def. 5), it follows that the family of images $U^* = \pi(U)$ of projection $\mathcal{U}^* = \{U^*\}$ forms a good cover of the orbit space $X^* = X/G$.

We have the following fact

Proposition ((7) Comparison of Definitions)

Let $\mathcal{U} = \{U_s\}$, split into orbits $\tilde{U}_{i\in I}$ be a good G-cover of X in the sense of Definition (5). Then it is a good G-cover of X in the sense of Definition (4).
Conversely, if $\mathcal{U} = \{U_s\}$, split into orbits $\tilde{U}_{i\in I}$ is a good G-cover of X in the sense of Definition (5) then it is a good G-cover of X in the sense of Definition (4).

Theorem (8)

If $\mathcal{U}$ is a locally finite, e.g. finite, equivariant good cover of a G-CW complex X, then $|\mathcal{N}(\mathcal{U})|$ of $\mathcal{N}(\mathcal{U})$ is G-homotopy equivalent to X.

Definition ((9) Strict covering and covering type)

By the definition, the strict G-covering type of a given space G-space X, denoted by $\mathrm{sct}_G(X)$ is the minimal cardinality of orbits an G-invariant regular good cover for X.
We define the G-covering type of a G-space X as the minimal value of $\mathrm{sct}_G(Y)$ of spaces Y that are G-homotopy equivalent to X:

$$\mathrm{ct}_G(X) := \min\{\mathrm{sct}_G(Y) \mid Y \overset{G}{\simeq} X\}$$

$\mathrm{sct}_G(X)$ can be ∞ (e.g., if X is an infinite discrete) or even undefined, if the space (e.g. the Hawaiian earring with the cyclic group C_2 permuting every consecutive pair of its loops). In what follows we will assume that the spaces admit finite good covers.

G-invariant regular open cover $\mathcal{U}$ of X induces an open good cover of the orbit space X/G as the projection map $\pi : X \to X/G$ is open and G-contraction of $\tilde{U}$ to an orbit Gx induces a contraction of $p(\tilde{U})$ to $* = [Gx]$ in X/G.

Corollary (10)

For a G-space X which is a G-CW complex we have

$$\mathrm{sct}(X/G) \leq \mathrm{sct}_G(X) \quad \textit{and respectively} \quad \mathrm{ct}(X/G) \leq \mathrm{ct}_G(X)$$

We end with a direct consequence of the Definition 5. $\Delta(K)$ the number of vertices of K and $\Delta^*(K)$ the number of orbits of vertices of K, i.e. the number of vertices of K/G.

Proposition (11)

We have

$$\mathrm{ct}_G(|K|) \leq \mathrm{sct}_G(|K|) \leq \Delta^*(K)$$

With complex K of dim d is associated a $d+1$-dimensional vector $\vec{f}(K) = (f_0(k), f_1(K), \dots f_d(K))$, where $f_i(K)$ is the number of i-dimensional simplices in K. If K is a G-complex of dimension d with a simplicial regular action of G, then we define

$$\vec{f}_G(K) := (f_{G,0}(K), f_{G,1}(K), \dots f_{G,d}(K)) \qquad (1)$$

where $f_{G,i}(K)$ is # of orbits of i-dim simplices of K. Note that the coordinates of classical vector

$$\vec{f}(K) := (f_0(K), f_1(K), \dots f_d(K))$$

where $f_i(K)$ is i-dimensional simplices of K are related to the corresponding coordinates of the $\vec{f}_G(K)$ by the formula

$$f_i(K) = \sum_{\sigma_i} |G/G_{\sigma_i}| = \sum_{1}^{f_{G,i}} |G/G_\sigma|,$$

where the sum is taken over representatives of all orbits of i-simplices σ of K or equivalently of all i-simplices of the induced triangulation of K/G.

The aim of this paper is to give some lower estimates of $f_{G,0}(K)$ and also $f_0(K)$.

Theorem (12)

Let X be a G-complex or more general G-CW complex. Assume that $\exists$ [!] minimal orbit type for the action on X, e.g. if the orbit types on X are ordered linearly $(H_1) \geq (H_2) \geq \cdots \geq (H_k)$. Then

$$\mathrm{ct}_G(X) \geq \frac{1}{2}\gamma_G(X)(\gamma_G(X)+1).$$

Remark (13)

The assumption of Theorem (12) is satisfied if the action is free or with one orbit type. Also $\forall$ G-space X if G is a group linearly ordered subgroups, e.g. if $G = \mathbb{Z}_{p^k}$ where p prime, and $k \geq 1$.

Example (14)

If we take $X = S(V)$, where V is $n+1$-dimensional complex , i.e. $2n+2$-dimensional real, free representation of $G = \mathbb{Z}_p$. Then $\gamma_G(S(V)) = \dim_{\mathbb{R}}(V) = 2n+2$ (cf. [Bartsch]) and $\mathrm{cat}_G(S(V)) = \dim_{\mathbb{R}}(V) = 2n+2$ (cf. [Marzantowicz]).
Consequently, if we substitute it to the formula of Theorem 12 we get

$$\mathrm{ct}_G(S(V)) \geq (n+1)\,(2n+3)\,,$$

Since here $\mathrm{ct}_G(S(V)) = \mathrm{ct}(S(V)/G) = \mathrm{ct}(L^{2n+1}(p))$ we get the same as estimate of $\mathrm{ct}(L^{2n+1}(p))$ as this given in [Govc, Marzantowicz, Pavešić 3] that is stronger than the previous of [Govc, Marzantowicz, Pavešić 1].

Definition (15)

The $(\mathcal{A}, K_G^*)$ – cup length of a pair (X, X') of G-spaces is the smallest r such that there exist $A_1, A_2, \dots, A_r \in \mathcal{A}$ and G-maps $\beta_i : A_i \to X$, $1 \le i \le r$ with the property that for all $\gamma \in K_G^*(X, X')$ and for all $\omega_i \in \ker \beta_i^*$ we have

$$\omega_1 \cup \omega_2 \cup \dots \cup \omega_r \cup \gamma = 0 \in K_G^*(X, X').$$

If there is not such r, we say that the $(\mathcal{A}, K_G^*)$ – cup length of (X, X') is ∞. $r = 0$ means that $K_G^*(X, X') = 0$. Moreover, the $(\mathcal{A}, K_G^*)$ – cup length of X is by definition the cup length of the pair $(X, \emptyset)$.

Taking $R := K_G(\mathrm{pt}) = R(G) \subset K_G^*(\mathrm{pt})$, we get

Definition (16)

The $(\mathcal{A}, K_G^*, R)$ – *length index* of a pair (X, X') of G-spaces is the smallest r such that there exist $A_1, A_2, \ldots, A_r \in \mathcal{A}$ with the following property:
For all $\gamma \in K_G^*(X, X')$ and all
$\omega_i \in R \cap \ker(K_G^*(\mathrm{pt}) \to K_G^*(A_i)) = \ker(K_G(\mathrm{pt}) \to K_G(A_i))$,
$i = 1, 2, \ldots, r$, the product $\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_r \cdot \gamma = 0 \in K_G^*(X, X')$.

From now till the end of this subsection we fix $G = \mathbb{Z}_{pn}$. After [Bartsch], for given two powers $1 \leq m \leq n \leq p^{k-1}$ of p we set

$$\mathcal{A}_{m,n} := \{G/H \,|\, H \subset G;\ m \leq |H| \leq n\}, \tag{2}$$

where $|H|$ is the cardinality of H. Next we put

$$\ell_n(X, X') = (\mathcal{A}_{m,n}, K_G^*, R) - \textit{length index of } \ (X, X'). \tag{3}$$

Theorem ((17) [Bartsch, Theorem 5.8])

Let V be an orth. repr. of $G = \mathbb{Z}_{p^k}$ with $V^G = \{0\}$ and $d = \dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} V$. Fix m, n two powers of p. Then

$$\ell_n(S(V)) \geq \begin{cases} 1 + \left[\frac{(d-1)m}{n}\right] & \text{if } \mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}, \\ \infty & \text{if } \mathcal{A}_{S(V)} \not\subset \mathcal{A}_{1,n}, \end{cases}$$

where $[x]$ denotes the least integer greater than or equal to x. Moreover, if $\mathcal{A}_{S(V)} \subset \mathcal{A}_{n,n}$, then $\ell_n(S(V)) = d$.

Theorem (18)

Let V be an orthogonal representation of $G = \mathbb{Z}_{p^k}$, and m, n, , d as in Theorem (17). If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ then

$$\mathrm{ct}_G(S(V)) \geq \frac{1}{2}\left(1 + \left[\frac{(d-1)m}{n}\right]\right)\left(2 + \left[\frac{(d-1)m}{n}\right]\right)$$

Note that if $k \geq 2$ then $S(V)$ and $m \neq n$ then $S(V)$ in Theorem (17) is not a G-space with one orbit type.

We estimate the G-cov. type in a bit more complicated situation.

Proposition (19)

Let $G = \mathbb{Z}_m$ be the cyclic group with $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, p_i prime. Let next, for each $1 \leq i \leq r$ V_i be an or. repr. of G given by a representation of $\mathbb{Z}_{p_i^{k_i}}$, denoted by V_i, and the projection from G onto $\mathbb{Z}_{p_i^{k_i}}$. Assume $V_i^G = \{0\}$ for all i. Then $\mathrm{ct}_G(S(V_1 \oplus V_2 \oplus \cdots \oplus V_r)) = \mathrm{ct}_{G_1}(S(V_1)) + \cdots \mathrm{ct}_{G_r}(S(V_r))$, where $G_i = \mathbb{Z}_{p^{k_i}}$ and $\mathrm{ct}_{G_i}(S(V_i))$ is estimated in Theorem 18.

Let W be an orth. r. of $G = \mathbb{Z}_m$ of dimension d such that the action of $G = \mathbb{Z}_m \subset S(\mathbb{C})$ rhe roots of unity is free on $S(W)$. Note that d odd if $m = 2$, otherwise d must be even. Let $V = W \oplus \mathbb{R}^1$. Then $S(V) = S(W) * S(\mathbb{R})$ and the action of G on $S(V)$ is free out of the poles. Then $\mathrm{ct}_G(S(V) \leq \mathrm{ct}(S(\mathbb{R})) + \mathrm{ct}_G(S(W))$ $= 2 + \mathrm{ct}_G(S(W))$. If dim $W = 2$ then $\mathrm{ct}_G(S(V) \leq 2 + 3 = 5$. If $d = 2$ then $\dim_{\mathbb{R}}(S(W)) = 1$ and consequently $\dim S(V) = 2$. Applying Theorem (29): $\mathrm{ct}_G(SV)) = \mathrm{ct}(S(V)/G) = \mathrm{ct}(\mathbb{S}^2) = 4$.

Let $h^*_G(\cdot)$ be a generalized equivariant cohomology theory. If X is a G–CW-complex, which is filtered by its skeletons $X^{(s)}$, we can define a filtration of $h^*_G(X)$ by setting
$h^*_{G,s}(X) := \ker(h^*_G(X) \to h^*_G(X^{(s-1)}))$.
The filtration of $h^*_G(X)$ defined above is decreasing:
$h^*_G(X) = h^*_{G,0}(X) \supset h^*_{G,1}(X) \supset \cdots h^*_{G,d-1}(X) \supset h^*_{G,d}(X) = 0$
where $d = \dim X$. And $h^*_G(X)$ is a filtered ring
$h^*_{G,s}(X) \cdot h^*_{G,s'}(X) \subset h^*_{G,s+s'}(X)$
Thus $h^*_{G,s}(X)$ is an ideal in $h^*_G(X)$. Also we have the following characterization of $h^*_{G,1}(X)$ (cf. [Segal] Proposition 5.1(i), page 146) $h^*_{G,1}(X) = \ker(h^*_G(X) \to \prod_{x\in X} h^*_G(G/G_x)) =$
$= \bigcap_{x\in X} \ker(h^*_G(X) \to h^*_G(G/G_x))$

Definition (20)

We say that an element u of $h^*_G(X)$ is of degree greater or equal to i, denoted by $|u| \geq i$, if $u \in h^*_{G,i}(X)$. We say that an element $u \in h^*_G(X)$ is of degree i if $|u| \geq i$, but $|u| \ngeq i+1$.

Theorem (21)

Let $u_1, \dots u_n \in h^*_G(X)$, $|u_k| \geq i_k$ *be such that* $u_1 \cdot u_2 \cdot \cdots u_n \neq 0 \in h^*_G(X)$. *Then* $\mathrm{ct}_G(i_1, \dots i_n) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+1)$.
If $i_1, \dots i_n$ *are not all equal, then* $\mathrm{ct}_G(X) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+2)$.

Lemma (22)

Let $X = U \cup V$ *where* $U, V \subset X$ *be open* G*-inv., and* u, $u \in \widetilde{h}^*_G(X)$ *be cohomology classes with* $u \cdot v \neq 0$. *If* U *is* G*-categorical in* X *then* $i^*_V(u)$ *is non-trivial in* $h^*_G(V)$ $(i_V : V \overset{G}{\hookrightarrow} X)$.

Lemma (23)

For $u \in h^*_G(X)$, *if* $|u| \geq i$ *then* $\mathrm{ct}_G(X) \geq i + 2$.

Theorem (21) doesn't require any condition on the orbits in X.

Let V be a complex representation of G of complex dim $n+1$ and $P(V)$ the projective space of V. The action of G on V induces an action on $P(V)$, since $g(\lambda v) = \lambda g(v)$ for $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$. Therefore ([Segal]) we have $K_G^0(P(V)) = \mathbb{R}(G)[\eta]/e(V)$, where $\mathbb{R}(G)$ is a representation ring of G and $e(V)$ is an ideal in $\mathbb{R}(G)$ generated by the element $\sum_{i=0}^{n} (-1)^i \wedge^i (V)\, \eta^{n+1-i}$. Here η is G-vector bundle conjugated to the G-Hopf bundle over $P(V)$. $K_G^1(P(V)) = 0$.

Theorem (24)

Let V be a complex representation of a finite group G of complex dimension $n+1$ and $P(V)$ the projective space of V. Then

$$\mathrm{ct}_G(P(V)) \geq (n+1)^2 .$$

The topological dimension $\dim P(V)$ is equal to $d = 2n$, i.e. $n = \frac{d}{2}$. Substituting it to the formula of Theorem (24) we get $\mathrm{ct}_G(P(V)) \geq \frac{(d+2)^2}{4}$, which express the estimate in term of the geometric dimension of $P(V)$.

Theorem (25)

Let $\mathbb{S}^n$ be an n-dimensional manifold being F_p-cohomology sphere on which acts the group $G = \mathbb{Z}_p^k$, p-prime, $k \geq 1$. Assume first that $\mathbb{S}^G = \emptyset$. Depending on p we have

$$\mathrm{ct}_G(\mathbb{S}^n) \geq \frac{(n+1)(n+2)}{2} \quad \text{if} \quad p = 2\,,$$

$$\mathrm{ct}_G(\mathbb{S}^n) \geq \frac{(d)(d+1)}{2} \quad \text{if} \quad p > 2, \quad \text{where} \quad d = \frac{n+1}{2} \quad \text{then}\,.$$

If $S^G \neq \emptyset$ then $S^G \underset{F_p}{\sim} \mathbb{S}^r$ is a F_p coh. sph. of dim. $r \geq 0$ and

$$\mathrm{sct}_G(\mathbb{S}^n) \geq (r+2) + \frac{(n-r-1)(n-r+2)}{2} \quad \text{if} \quad p = 2\,,$$

$$\mathrm{ct}_G(\mathbb{S}^n) \geq (r+2) + \frac{(d-1)(d+1)}{2} \quad \text{if} \quad p > 2, \quad \text{where} \quad d = \frac{n-r}{2}\,.$$

Let Σ_g be oriented surface of genus $g \geq 0$. Suppose that G acts **effectively** on Σ_g **preserving orientation** , i.e. it is a subgroup of $\mathrm{Homeo}^+(\Sigma_g)$. It is know (Hurwitz for Σ_g with $g \geq 1$, Brouwer, Kerekjarto and Eilenberg for $\Sigma_g = S^2$ and a folklore for $\Sigma_g = \mathbb{T}^2$)) that there exists a holomorphic structure $\mathcal{H}$ on Σ_g in which $\mathrm{Homeo}^+(\Sigma_g)$ is equal to the group of biholomorphic isomorphisms $\mathrm{Hol}(\Sigma_g, \mathcal{H})$ of $(\Sigma, \mathcal{H})$. More precisely we have the following

Theorem ((26) Geometrization of action)

Given a finite group G of orientation-preserving homeomorphisms of a compact surface of an arbitrary genus g, there is a complex structure on X with respect to which G is a subgroup of the group Hol of all its conformal maps. Furthermore, the orbit space $X' = X/G$ is a compact surface of genus $g' < g$. Moreover the relation between g and g' is given by the Riemann-Hurwitz formula (4).

Moreover, Hurwitz' theorem says that the order of $\mathrm{Hol}(\Sigma_g, \mathcal{H})$ is $\leq 84(g-1)$ if $g \geq 2$.

Let Σ_g be a compact surface of genus $g > 1$ and let G be a group of holomorphic automorphisms of Σ_g. Let $\Sigma_{g'} = \Sigma_g/G$ be the quotient surface of genus g' with the projection $\pi : X \to X'$ and let $\{x'_1, \ldots, x'_r\}$ be the set of all points over which π is branched. Denote by $\mathcal{S}$ the set of images of singular orbits $\{x'_1, \ldots, x'_r\}$ in Σ'.
Riemann-Hurwitz formula:

$$g = 1 + m(g' - 1) + \frac{1}{2}\, m \sum_{j=1}^{r} (1 - \frac{1}{m_j}), \qquad (4)$$

which let us also express g' as a function of g.
We have a classical result which is converse to the Riemann-Hurwitz formula (see [Broughton, Proposition 2.1]).

Proposition ((27) Riemann's Existence Theorem)

The group G acts on the surface Σ_g, of genus g, with branching data $(g', r, m_1, \ldots, m_r,)$ if and only if the Riemann-Hurwitz equation (4) above is satisfied, and G has a generating $(g' \ : m_1, \ldots, m_r)$-vector.

We shall use the fact for any closed surface, also non-oriented, the number of vertices of minimal triangulation is given by

Theorem ((28) Jungerman and Ringel)

Let Σ_g be a closed surface different from the orientable surface of genus 2 (M2), the Klein bottle (N2) and the non-orientable surface of genus 3 (N3). There exists a triangulation of S_g with n vertices if and only if

$$(+) \qquad n \geq n_g = \left\lceil \frac{7+\sqrt{49-24\chi(\Sigma_g)}}{2} \right\rceil .$$

For (M2), (N2) and (N3), n_g is replaced by $n_g + 1$ in this formula.

Here $\lceil \alpha \rceil$ means the ceiling of a real number α.

Theorem (29)

Let Σ_g be an oriented surface of genus g. Suppose that a finite group G acts on Σ_g preserving orientation and $\Sigma_{g'} = \Sigma_g / G$ is the quotient surface. Let $r = |\mathcal{S}|$ be the number of singular fibers of the projection $\pi : \Sigma_g \to \Sigma_{g'}$, $n_{g'}$ be the number defined in Theorem 28, and $n^1_{g'}$ the number defined above.
Then for the number of orbits of minimal regular G-triangulation K of Σ_g we have the estimate

$$n_{g'} \leq f_{G,0}(K) \leq n_{g'} + n^1_{g'}(r)\,,$$
$$n_{g'} \leq f_{G,0}(K) \leq r + n^1_{g'}(r) \;\; \text{if} \;\; r > n_{g'}\,.$$

And the number of its vertices is estimated by

$$\sum_{j=1}^r \frac{m}{m_j} + (n_{g'} - r)\,m \leq f_0(K) \leq \sum_{j=1}^r \frac{m}{m_j} + (n_{g'} - r + n^1_{g'}(r))\,m,$$
$$\text{if} \;\; r \leq n_{g'}\,, \quad \text{or}$$
$$\sum_{j=1}^r \frac{m}{m_j} \leq f_0(K) \leq \sum_{j=1}^r \frac{m}{m_j} + n^1_{g'}(r)\,m \;\; \text{if} \;\; r > n_{g'}\,.$$

Let $X = |K|$ be the body of one-dimensional simplicial connected complex, i.e. a finite graph, with a regular simplicial action of the group G. It means that G permutes vertices and edges of K and $g[v_1, v_2] \subset [v_1, v_2]$ implies that $g = \mathrm{id}_{[v_1,v_2]}$ for every edge $e = [v_1, v_2]$.
Before the discussion let us remind the corresponding result for the not equivariant case, i.e. when there is not action of G, or equivalently $G = e$ (cf. [Karoubi, Weibel, Proposition 4.1]
If X_h is a bouquet of $h > 0$ circles then

$$\mathrm{ct}(X_h) = \left\lceil \frac{3+\sqrt{1+8h}}{2} \right\rceil \tag{5}$$

That is, $\mathrm{ct}(X_h)$ is the unique integer n such that

$$\binom{n-2}{2} < h \leq \binom{n-1}{2} \tag{6}$$

If X is as above and $e_1, \ldots e_p$ be all edges outgoing (or equivalently ingoing) from $\{*\}$ then by $X'_{(H)}$ we denote $X \setminus N_\epsilon(\{*\})$, where $N_\epsilon(A)$ is a open and invariant neighbourhood of invariant set A. Let next $0 \leq h_{(H)}(X)$, shortly $h_{(H)}$, be the number of loops of $X'_{(H)}/G$, i.e.the number of generators of $\pi_1(X'_{(H)}/G)$.

By $X'_{(H)}$ we denote the compact closed set (graph) $X \setminus N_\epsilon(X^{(K)\not\succ(H)})$. Next $0 \leq h_{(H)}(X)$, shortly $h_{(H)}$, be the number of loops of $X'_{(H)}/G$, i.e.the number of generators of $\pi_1(X'_{(H)}/G)$.

Definition

Let X be a finite regular G-graph and (H) an orbit type such that $X_{(H)} \neq \emptyset$. We say that (H) is essential in X if for every regular G-graph K, $X \overset{G}{\sim} K$ such that $f_0(K) = \mathrm{ct}_G(X)$, there exists a vertex $v \in K$ with the isotropy group $G_v = H$.
Otherwise we call a nonempty $X_{(H)} \neq \emptyset$ orbit type nonessential.

For a connected component of an essential orbit type $X^*_{(H),i)}$ of $X^*_{(H)} = X_{(H)}/G$, $1 \leq i \leq c(H)$ we put $\hat{\mathrm{ct}}(X^*_{(H),i}) = \left\lceil \frac{3+\sqrt{1+8h_{(H)}(i)}}{2} \right\rceil$ if we have nontrivial loops in $(X^*_{(H)})_{(H),i}$, or $\hat{\mathrm{ct}}((X^*_{(H)})_{(H),i}) = 1$ otherwise, i.e. if $(X^*_{(H)})_{(H),i}$ is a tree. If $X^*_{(H),i)}$ is nonessential we put $\hat{\mathrm{ct}}((X^*_{(H)})_{(H),i}) = 0$.

Theorem (31)

Let X be a finite connected graph with a regular simplicial action of a finite group G. Let next $\mathcal{S}_G(X)$ be the subset of the set of all orbit types $\mathcal{S}_G$ consisting of all (H) such that $X_{(H)} \neq \emptyset$. Then

$$\mathrm{ct}_G(X) = \sum_{(H)\in\mathcal{S}_G(X)} \sum_{i=1}^{c(H)} \hat{\mathrm{ct}}((X^*_{(H)})_{(H),i}),$$


GOODBYE!

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