Some problems of TC and category-like invariants and groups

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Some problems of Applied and Computational Topology
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Homotopical invariants of groups

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With this in mind, any homotopy-type invariant Γ of topological spaces allows us to define an isomorphism-type invariant of groups just defining $\Gamma(G) = \Gamma(BG)$.

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Fundamental question

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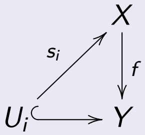
This question has no easy answer. The homotopy type of BG (and hence all of its homotopy-type invariants) is completely determined by G BUT the description of the invariant may involve homotopy-theoretical constructions that can not be expressed in terms of classifying spaces.

The setting: sectional category

Sectional category of a map

Given a continuous map $f : X \to Y$ we define its sectional category $\mathrm{secat}(f)$ as the least integer $n \geq 0$ such that there exists an open cover of Y $\{U_i\}_{0 \leq i \leq n+1}$ admitting an homotopy section for every i, $s_i : U_i \to X$



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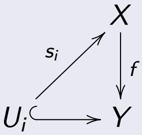

If X is contractible, then $\mathrm{secat}(f)$ becomes the well-known Lusternik-Schnirelmann category of Y.

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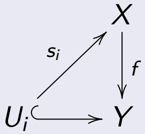

If f corresponds to the path space fibration, i.e

$$f : PX \to X \times X$$

assigning $\gamma \mapsto (\gamma(0), \gamma(1))$ then secat(f) coincides with the Topological Complexity of X, TC(X).

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2. Let $k > 0$ the maximal integer such that there exist

$$u_1, \cdots, u_k \in \ker\{\tilde{H}^*(B, R) \xrightarrow{p^*} \tilde{H}^*(E, R)\}$$

with $u_1 \smile \cdots \smile u_k \neq 0$. Then $\operatorname{secat}(p) \geq k$.

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Theorem [Eilenberg-Ganea, 1957]

Let G be a group. If $cat(G) \geq 3$ then

$$cat(G) = cd(G) = gd(G)$$

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The answer is currently unknown.

Sectional category of subgroup inclusions

Sectional category of group morphisms

Given a group homomorphism $f : H \to G$ we define $\mathrm{secat}(f)$ as the sectional category of the induced map $Bf : BH \to BG$

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Sectional category of subgroup inclusions

A characterization

We have $\mathrm{secat}(H \hookrightarrow G) \leq n$ if and only if the Borel fibration

$$p_n : EG \times_G *^{n+1}(G/H) \to EG/G$$

has a section (i.e $\mathrm{secat}(p_n) = 0$).

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$$\begin{array}{ccc} EG & \xrightarrow{\rho} & E_{\langle H \rangle} G \\ & \searrow \quad \nearrow & \\ & (E_{\langle H \rangle} G)_n & \end{array}$$

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- Given a G-module M, define $\mathrm{Div}^*(M) = \ker\{H^*(G, M) \to H^*(H, \mathrm{res}^G_H M)\}$. If there exists $\{x_i\}_{1\leq i\leq k}$ with $x_i \in \mathrm{Div}^*(M)$ such that $x_1 \smile \cdots \smile x_k \neq 0$ then $\mathrm{secat}(H \hookrightarrow G) \geq k$.

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  The temptation here is then to use some relative cohomology.

Relative Bernstein class

Splicing together short exact sequences of G-modules

$$0 \to K^{\otimes n+1} \to \mathbb{Z}[G] \otimes K^{\otimes n} \xrightarrow{\varepsilon \otimes \mathrm{id}} K^{\otimes n} \to 0$$

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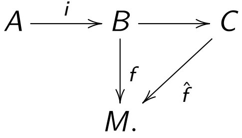
Given a short exact sequence of G-modules

$$0 \to A \xrightarrow{i} B \to C \to 0,$$

and a G-module map $f\colon B \to M$ with $f \circ i = 0$, we will write $\hat{f}$ for the induced map



Relative Bernstein class

Proposition

Let $[a] \in H^p(G, A)$ and $[b] \in H^q(G, B)$ be cohomology classes represented by cocycles $a\colon \mathbb{Z}[G] \otimes K^{\otimes p} \to A$ and $b\colon \mathbb{Z}[G] \otimes K^{\otimes q} \to B$. Then the cup product $[a][b] \in H^{p+q}(G, A \otimes B)$ is represented by the map

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$\omega \in H^1(G, I)$ is the class represented by a G-module homomorphism

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Bernstein relative class

$\omega \in H^1(G, I)$ is the class represented by a G-module homomorphism

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defined as the composition of $\varepsilon \otimes \mathrm{id}$ and the map $\mu\colon K \to I$ induced by the canonical projection $G \to G/H$

Relative Bernstein class

Powers of the relative Bernstein class

By cup product description, the n-th power $\omega^n \in H^n(G, I^{\otimes n})$ is represented by the map

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Proposition (Generalized Costa-Farber)

If $n = \mathrm{cd}\, G \geq 3$, then $\mathrm{secat}(H \hookrightarrow G) \leq n - 1$ if and only if $\omega^n = 0$.

Adamson relative cohomology

$(G : H)$-exactness

Given $H \leq G$, an exact sequence of G-modules

$$\cdots \to M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \to \cdots$$

is said to be $(G : H)$-exact if $M_i \cong \ker f_i \oplus N_i$ as H-module, for every i.

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Cup product in Adamson cohomology and canonical class

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Cup product

If $\alpha_i = [a_i]$ we define the cup product $\alpha_1 \smile \alpha_2$ as the class represented by the composition

$$\mathbb{Z}[G/H] \otimes I^{\otimes(n_1+n_2)} \xrightarrow{\varepsilon \otimes \mathrm{Id}} I^{\otimes(n_1+n_2)} \xrightarrow{\widehat{a_1} \otimes \widehat{a_2}} M \otimes M$$

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The class ϕ is universal in the sense that for every other $\lambda \in H^n([G:H], M)$ there exists a G-morphism $f : I^{\otimes n} \to M$ such that $\lambda = f^*(\phi^n)$.

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As a corollary, $cd(G:H) = \mathrm{height}(\phi) = \max\{n | \overbrace{\phi \smile \cdots \smile \phi}^{n} \neq 0\}$.

Adamson cohomology and zero divisors

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$$\ker\left[H^1(G, \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes n-1}, M)) \to H^1(H, \mathrm{Hom}_{\mathbb{Z}}(I^{\otimes n-1}, M))\right]$$

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In particular, there exists $\rho^* : H^*([G:H], M) \to H^*(G, M)$ inducing $H^1([G:H], M) = \mathrm{Div}^1(M)$ as above, but $H^n([G:H], M)$ can only be identified with $\mathrm{Div}^n(M)$ under very restrictive hypothesis.

A spectral sequence

Take the (G, H)-projective resolution of $\mathbb{Z}$

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$$\cdots \to \mathbb{Z}[G/H] \otimes I^{\otimes n} \to \cdots \to \mathbb{Z}[G/H] \otimes I \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0.$$

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Proposition

There exists a spectral sequence

$$\mathrm{E}_2^{p,q} = H^p(\mathrm{Ext}_G^q(\mathbb{Z}[G/H] \otimes I^{\otimes *}, M)) \Rightarrow H^{p+q}(G, M)$$

such that $\mathrm{E}_2^{p,0} = H^p([G : H], M)$.

Some results

A lower bound for secat$(H \hookrightarrow G)$

Let M be a G-module such that $\rho^* : H^n([G : H], M) \to H^*(G, M)$ is non trivial. Then secat$(H \hookrightarrow G) \geq n$.

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Small corollary

Let $n > \mathrm{secat}(H \hookrightarrow G)$ and M a G-module such that $\mathrm{Div}^n(M) \neq 0$. Then there exists a subgroup $U \leq H$ and an integer $m < n$ such that $H^m(U, M) \neq 0$.

What to do now?

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- Given any n, it is easy to construct examples such that $cd(G : H) - \text{secat}(H \hookrightarrow G) = n$.

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Example

If $G = A *_C A$ amalgam for $C \trianglelefteq A$ then

$$\mathrm{cd}_{\mathcal{G}}(G \rtimes G) = \mathrm{cd}_{\langle \Delta_G \rangle}(G \times G) \geq \mathrm{cd}(A/C).$$

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Essential classes

Farber and Mescher defined essential cohomology classes in $H^*(\pi \times \pi, M)$.

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$$\mu^*(\mathbf{v}^n) = \alpha$$

where I denotes the augmentation ideal and $\mathbf{v}$ denotes the 1-dimensional *Bernstein canonical class*, represented by the short exact sequence of $(\pi \times \pi)$-modules

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Relative essential classes

Let (G, H) a pair group, we say that $\alpha \in H^*(G, M)$ is *essential relative to H* if there exists a homomorphism of $\mathbb{Z}[G]$-modules $\mu \colon I^n \to M$ such that

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Essential classes

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The *essential dimension* of a pair (G,H) (denoted by $\rho^*_{[G:H]}$) as the greatest integer $n \geq 0$ such that the canonical homomorphism in cohomology defined above

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How much can be read from the spectral sequence in Adamson and $\ker \rho^*$?

TC of $K(G,1)$-spaces: $\mathcal{A}$-genus

$\mathcal{A}$-genus (Clapp, Puppe)

G group, X a G-space and $\mathcal{A}$ a family of G-spaces. The $\mathcal{A}$-genus of X, is the smallest integer $r \geq 0$ such that there exists a G-invariant covering $X_0, \cdots, X_r$ of X having the property that for every $i = 0, \cdots, r$ there exists $A_i \in \mathcal{A}$ and a G-equivariant map $X_i \to A_i$.

TC of $K(G,1)$-spaces: $\mathcal{A}$-genus

Proposition

Let X be a connected CW-complex. If $q : \hat{X} \to X$ is a connected covering, then $\mathrm{secat}(q) = \mathcal{A}\text{-}\mathrm{genus}(\tilde{X})$ where $\mathcal{A} = \{\pi_1(X)/\pi_1(\hat{X})\}$.

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Corollary (TC as $\mathcal{A}$-genus)

Let X be a connected CW-complex with $\pi_1(X) = G$ and $\mathcal{A} = \{(G \times G)/\Delta G\}$. If X is aspherical, then $\mathrm{TC}(X) = \mathcal{A}\text{-genus}(\tilde{X} \times \tilde{X})$.

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- $\mathrm{TC}(G) \leq \mathcal{A} - \mathrm{genus}(E_{\mathcal{F}}(G \times G))$.

TC of $K(G,1)$-spaces: $\mathcal{A}$-genus

We can recover the classical properties of TC in terms of $\mathcal{A}-$genus and also derive some new boundaries, like

- $\mathrm{TC}(G) \leq \mathcal{A} - \mathrm{genus}(E_{\mathcal{F}}(G \times G))$.
- $\mathrm{TC}(G) \leq \mathrm{cd}_{\langle H \rangle} G \times G$ for any $H \subset G \times G$ subconjugate to ΔG.

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For any H overgroup of ΔG take $\mathcal{B} = \{(G \times G)/H\}$.

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$$\mathcal{B}\text{-genus}(E(G \times G)) \leq \mathcal{A}\text{-genus}(E(G \times G)) = \mathrm{TC}(G).$$

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Proposition

There is a one to one correspondence between overgroups H of ΔG in $G \times G$ and normal subgroups $N \trianglelefteq G$.

$\mathcal{A}$-genus and proper category-like invariants

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Theorem [Leary-Nucinkis, 01]

For any connected CW-complex X there exists a discrete group G_X and a contractible proper $G_X - CW$-complex E_X such that E_X/G_X is homotopy equivalent to X.

$\mathcal{A}$-genus and proper category-like invariants

Definition

Let G an arbitrary group, and $\mathcal{F}in$ the closed family of finite subgroups. Define the proper genus of the group, G $\underline{\mathrm{genus}}(G)$, as $\mathcal{F}in$-$\mathrm{genus}(\underline{E}G)$.

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G be a discrete group with torsion such that $\underline{B}G$ has dimension n and $H^n(\underline{B}G, M)$ for some G-module M. Then, we have

$$\dim_G(\underline{E}G) = \mathrm{cd}_{\mathcal{F}in}(G) \leq \underline{\mathrm{genus}}(G)$$

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As a consequence the difference between $\underline{\mathrm{genus}}$ and $\underline{\mathrm{TC}}$ may be arbitrarily great.

Symmetries in the configuration spaces

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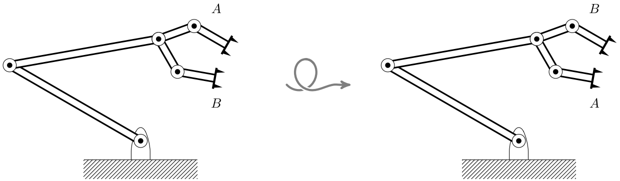

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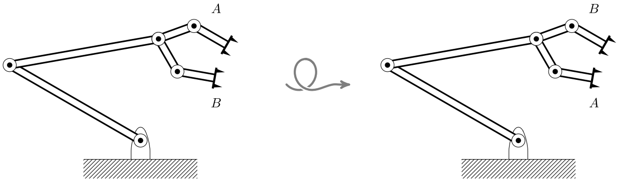

The original topological complexity does not take this sort of phenomena into account!

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Equivariant TC

$\mathrm{TC}_G(X)$ is the minimal number $k \geq 0$ such that there exists an open G-invariant cover of $X \times X$ by $k-1$ sets which admit G-equivariant motion planners.

Strongly equivariant TC

$\mathrm{TC}^*_G(X)$ is defined as TC_G but using the component-wise $(G \times G)$ action.

Symmetries in the configuration spaces

$PX \times_{X/G} PX = (\gamma, \delta) \in PX \times PX | G\gamma(1) = G\delta(0)$ as a $(G \times G)$-space component-wise.

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$PX \times_{X/G} PX = (\gamma, \delta) \in PX \times PX | G\gamma(1) = G\delta(0)$ as a $(G \times G)$-space component-wise. The map $\pi_2 \colon PX \times_{X/G} PX \to X \times X$ given by $\pi_2(\gamma, \delta) = (\gamma(0), \delta(1))$ is a $(G \times G)$-fibration.

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Invariant TC

$\mathrm{TC}^G(X)$ is defined as the least integer $k \geq 0$ such that there exists an open $(G \times G)$-invariant cover of $k - 1$ sets with local homotopy sections of π_2.

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- $\mathrm{TC}(X) \leq \mathrm{TC}_G(X) \leq \mathrm{TC}^*_G(X)$.
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All of them focus on motion planning algorithms that are symmetric. What about using the symmetries already present to simplify the task of motion planning?

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$$\mathcal{P}_k(X) = \{(\gamma_1, \cdots, \gamma_k) \in (PX)^k | G\gamma_i(1) = G\gamma_{i+1}(0)\}.$$

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This is a fibration:

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$\mathrm{TC}^{G,k}(X) = \mathrm{secat}(\pi_k).$

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- What could be done with MV spectral sequence?

¡Gracias por su atención!
Thank you for your attention!
Dziękuję za uwagę!