Random Simple-Homotopy Theory

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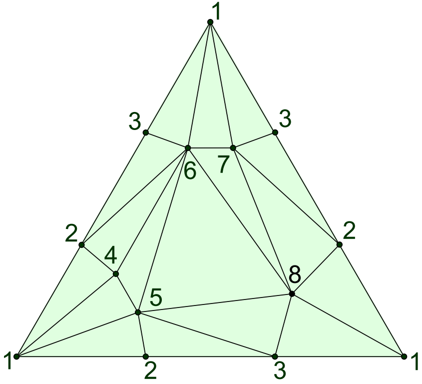

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- Reduce the size of complexes
- Identify substructures in complexes
- Test contractibility

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- Collapses.
- Discrete Morse theory.
- Collapses and anti-collapses (simple-homotopy).

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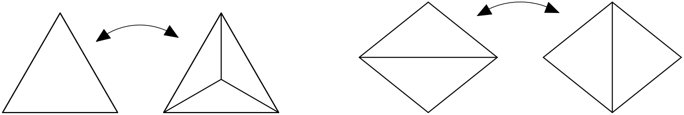
New approach:

- Collapses and (few) elementary expansions.

Bistellar flips



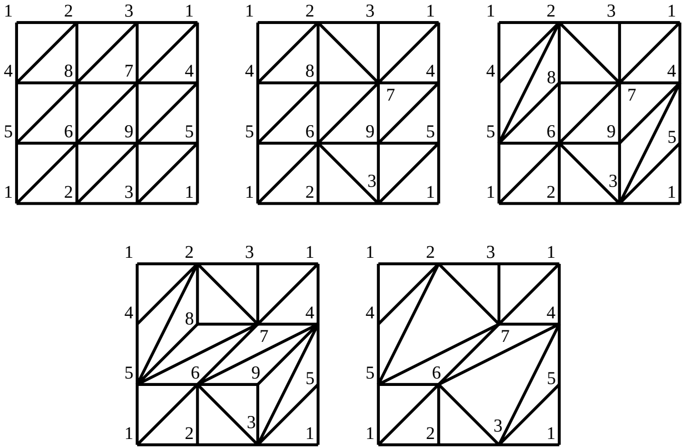
Local modifications of the triangulation by cutting out
a (triangulated) ball and replacing it by a re-triangulated ball.

[Pachner, 1986]

Two combinatorial triangulations of a d-manifold are bistellarly equivalent if and only if they are PL homeomorphic.

Bistellar flips on the torus

9-vertex torus



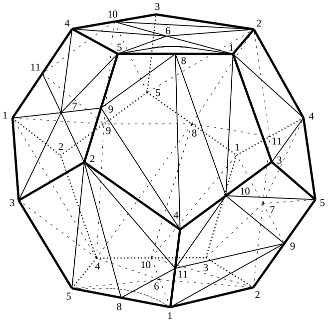
Möbius' 7-vertex torus

Simulated annealing approach

[Björner, L., 2000]

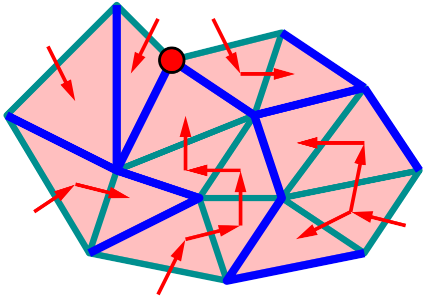
“Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere.”

$f = (16, 106, 180, 90)$



Collapses

- i-face is free, if it is contained in a unique $(i+1)$-face.
- Collapsing step: delete pair.
- Complex K is collapsible, if it can be collapsed to a point.



Random discrete Morse theory

[Benedetti, L., 2014]

"Random discrete Morse theory and a new library of triangulations."

- Pick free faces uniformly at random.
- Pick facet as critical face if stuck.
- Rerun.

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Discrete Morse vector: $c = (c_0, c_1, \ldots, c_d)$

$c_i =$ # critical i-faces

[Whitehead, 1939; Forman, 1998, 2002]

A combinatorial d-manifold is a PL d-sphere if and only if it admits some subdivision with a spherical discrete Morse vector $(1, 0, \ldots, 0, 1)$.

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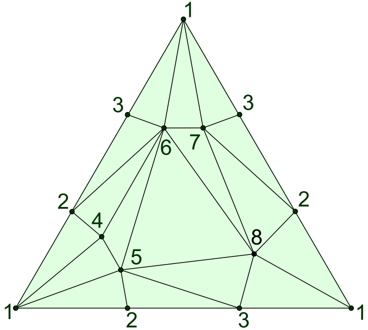
[Adiprasito, Benedetti, L, 2017]

"Extremal examples of collapsible complexes and random discrete Morse theory."

Example of a non-PL 5-manifold, with face vector $f = (5013, 72300, 290944, 495912, 383136, 110880)$, that is collapsible, but not homeomorphic to a ball.

How can we recognize that a complex is contractible?



Dunce hat (here triangulated with 8 vertices) is contractible,
but not collapsible. [Zeeman, 1963]

Simple-homotopy theory

[Whitehead, 1939]

"Simplicial Spaces, Nuclei and m-Groups."

- Allow i-collapses and i-anti-collapses.
- Two simplicial complexes are simple-homotopy equivalent if they can be connected by a sequence of i-collapses and i-anti-collapses.
- Simple-homotopy equivalent implies homotopy equivalent.

  (The two notions coincide for complexes with trivial Whitehead group, in particular, for complexes with trivial fundamental group.)

Anti-collapses are problematic

For every i-anti-collapse that adds an i-simplex to a complex, all the $(i-1)$-faces of the i-simplex, but one, already have to be present.

This often requires to add low-dimensional faces first before an i-simplex can be added.

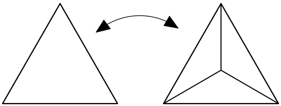
Pure elementary expansions

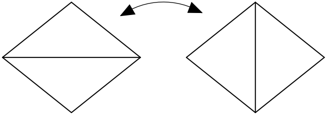
[Benedetti, Lai, Lofano, L., 2021+]

Definition

Let K be a d-dimensional complex.

A pure elementary expansion (of dimension $d+1$) is the gluing of a $(d+1)$-simplex σ to K along an induced pure d-ball on the vertex-set of σ.





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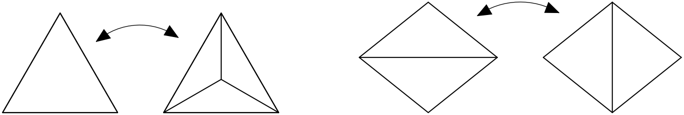

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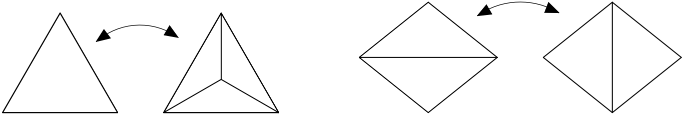

- Bistellar flips can be expressed as pure elementary expansions followed by collapses.
- Every pure elementary expansion can be expressed as a sequence of i-anti-collapses (possibly of different dimensions).

Algorithm: Random simple-homotopy (RSHT)

[Benedetti, Lai, Lofano L., 2021+]

Input: simplicial complex K
Output: modified simplicial complex
while $\dim(K) \neq 0$ ***and*** $i < max\_step$ **do**
  while K *has free faces* **do**
    perform a random elementary collapse
  end
  if $\dim(K) = d \neq 0$ **then**
    perform a single random pure elementary $(d+1)$-expansion
    [perform an elementary collapse deleting the newly added $(d+1)$-dimensional face and one of its d-faces that was already in K]
  end
  i++
end
return K

Reduction of (d+1)-expansions to bistellar flips

[Bagchi, Datta, 2005]

Every contractible simplicial complex with $n \leq 7$ vertices is collapsible.

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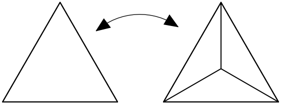
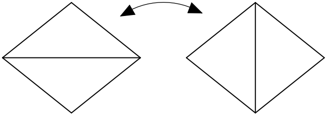
Let K be a triangulated d-manifold with $d \leq 6$.

Then any pure elementary $(d+1)$-expansion followed by collapses (as long as free faces are available) induces a bistellar flip on K.

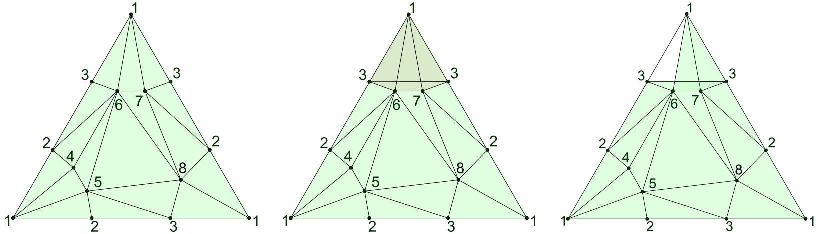




Manifold stability

[Benedetti, Lai, Lofano L., 2021+]

Let K be a (not necessarily pure) simplicial complex. If we run RSHT on K and at some point reach a simplicial complex K' that triangulates a d-manifold with $d \leq 6$, then from then on, whenever there are no free faces in the further run of RSHT, the respective temporary complex $\tilde{K}$ is a d-manifold as well, and $\tilde{K}$ is bistellarly equivalent to K'.



(a) Dunce Hat with 8 vertices.

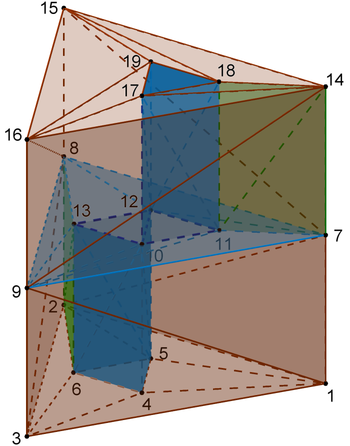
(b) Anticollapsing the tetrahedron 1367.

(c) Collapsing the tetrahedron 1367.

Triangulations of the dunce hat are examples that are contractible, but not collapsible.

The addition of a single tetrahedron makes the 8-vertex triangulation of the dunce hat collapsible.

Bing's house with two rooms



First add five tetrahedra 7 9 11 14, 11 14 17 18, 7 11 15 17, 7 10 11 17 and 7 14 15 17 . . .

Contractible non-collapsible complexes

complex	f-vector	rounds	# added tets (minimum)	# added tets (mean)
Dunce hat	$(8, 24, 17)$	10^4	1	2.41
Abalone	$(15, 50, 36)$	10^4	3	32.42
Bing's House	$(19, 65, 47)$	10^4	7	58.10
BH(3)	$(43, 150, 108)$	10^4	19	147.97
BH(4)	$(57, 200, 144)$	10^4	29	167.77
BH(5)	$(71, 250, 180)$	10^4	27	195.89
BH(6)	$(85, 300, 216)$	10^4	34	221.26
BH(7)	$(99, 350, 252)$	10^4	41	244.58
`Two_optima`	$(106, 596, 1064, 573)$	10^3	1	7.05
Furch's knotted ball	$(380, 1929, 2722, 1172)$	10^3	1459	1949.95
`double_trefoil_ball`	$(15, 93, 145, 66)$	10^3	1	29.60
`triple_trefoil_arc`	$(17, 127, 208, 97)$	10^3	6	94.68

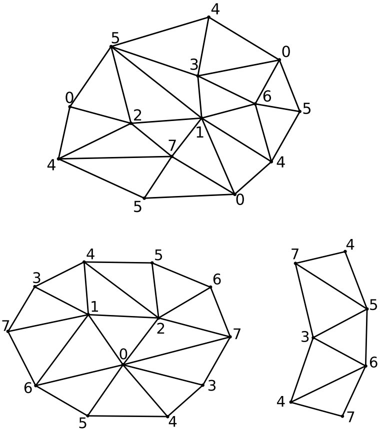
Substructure identification

- $\mathbb{R}P^3$ (11 vertices) to $\mathbb{R}P^2$ (6 vertices) (25.25 expansions).
- $\mathbb{R}P^4$ (16 vertices) to $\mathbb{R}P^3$ (11 vertices) (885.60 expansions).
- $S^2 \times S^1$ (100 · 10 vertices) to $S^2 \vee S^1$ (4 + 25.8 vertices) (1108.23 expansions).
- $S^3 \times S^2 \times S^1$ to $S^3 \vee S^2 \vee S^1$ ...

Remove a random facet from a manifold and then simplify!

2-complexes with torsion

Starting from lens spaces $L(p, 1)$ ...



torsion p	f-vector
2	$(6, 15, 10)$
3	$(8, 24, 17)$
4	$(8, 26, 19)$
5	$(9, 32, 24)$
6	$(9, 33, 25)$
7	$(9, 34, 26)$
8	$(9, 35, 27)$
9	$(9, 36, 28)$
10	$(9, 36, 28)$
11	$(10, 42, 33)$
12	$(10, 42, 33)$
13	$(10, 43, 34)$

Substructure identification / surface reconstruction

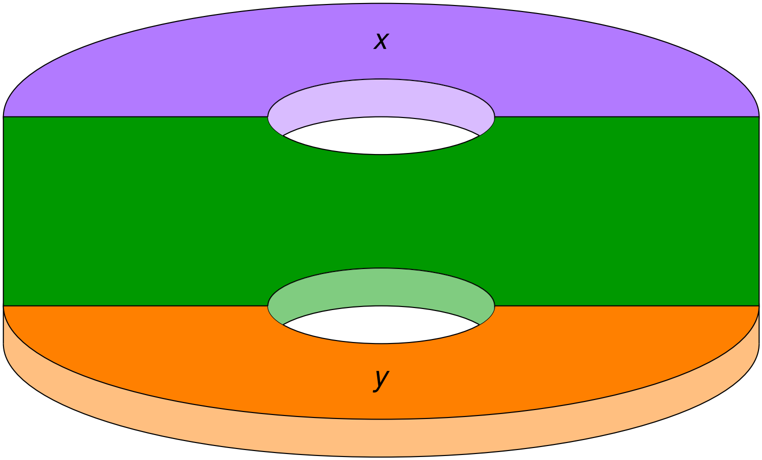
3-complex	f-vector of 3-complex	final f-vector
$T^2 \times I$	$(77, 511, 854, 420)$	$(7, 21, 14)$
$M(2,+) \times I$	$(121, 929, 1586, 780)$	$[(9, 32, 24)]$??
$M(5,+) \times I$	$(253, 2183, 3782, 1860)$	$(12, 60, 40)$
$M(6,+) \times I$	$(297, 2601, 4514, 2220)$	$(13, 69, 46)$
$M(10,+) \times I$	$(473, 4273, 7442, 3660)$	$(18, 108, 72)$

- take connected sums of the torus T^2
- add 100 vertices
- run 200,000 bistellar edge flips
- take cross product with path of length 10
- simplify

Surface triangulation is recovered in each run!

Limitations

Akbulut–Kirby 4-spheres



Balanced presentation of the trivial group:

$$G := < x, y \mid x, y >$$

Akbulut–Kirby 4-spheres

Nontrivial balanced presentation of the trivial group:

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$$\begin{aligned} y^r &= x^{-1}y^{-1}x^r yx \\ &= x^{-1}y^{-1}y^{r-1}yx \\ &= x^{-1}y^{r-1}x \\ &= x^r \end{aligned}$$

Akbulut–Kirby 4-spheres

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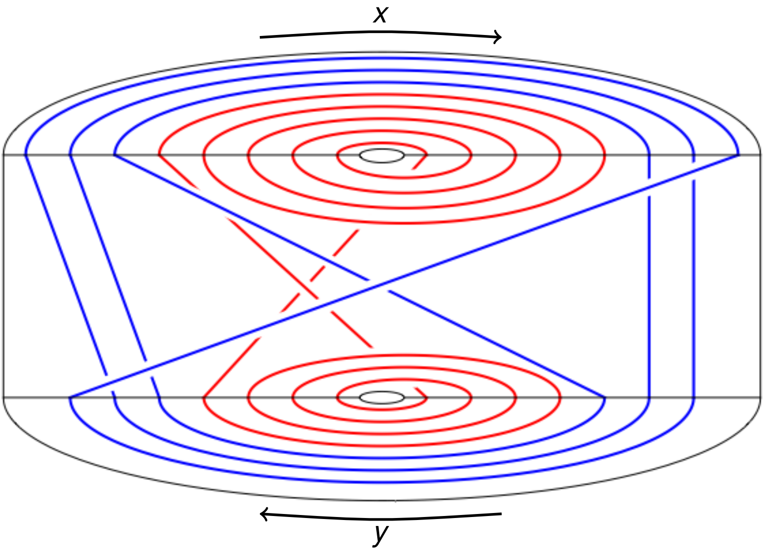
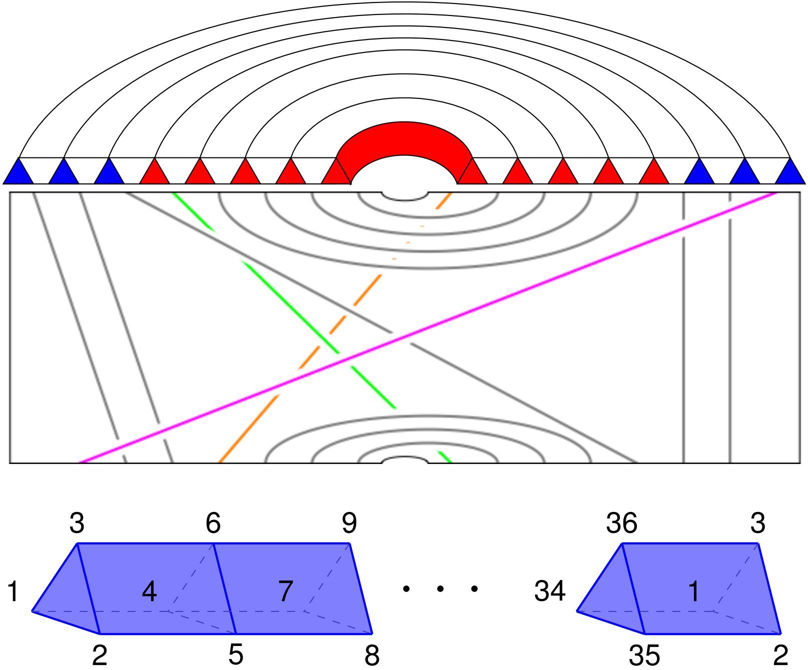
Akbulut–Kirby 4-spheres are defined via these presentations.



$x^5 = y^4, \quad xyx = yxy$


3
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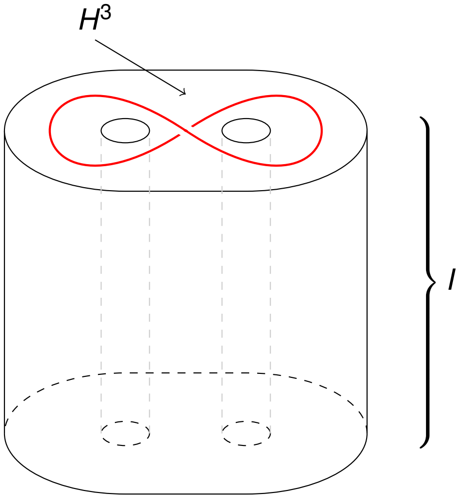

H^3
I

[Tsuruga, L., 2014]

The Akbulut–Kirby 4-spheres can be triangulated with face vector

$$f = (\ 176 + 64r, 2390 + 1120r, 7820 + 3840r,\\ 9340 + 4640r, 3736 + 1856r\ )$$

for $r \geq 3$.

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$$\begin{aligned}
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r = 6:&\quad f = (560,\ \ 9110, 30860, 37180, 14872),\\
r = 7:&\quad f = (624, 10230, 34700, 41820, 16728),\\
r = 8:&\quad f = (688, 11350, 38540, 46460, 18584),\\
r = 9:&\quad f = (752, 12470, 42380, 51100, 20440),\\
r = 10:&\quad f = (816, 13590, 46220, 55740, 22296),\\
\dots & \qquad\qquad \dots \qquad\qquad \dots
\end{aligned}$$

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\end{aligned}$$

Reduction with bistellar flips to 23+ vertices.

[Akbulut, 2010]

The Akbulut–Kirby 4-spheres are standard PL 4-spheres.

They are the only known explicit examples of triangulated spheres for which we fail to recognize them heuristically.

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It is open,

- whether all 4-spheres (obtained via balanced presentations of the trivial group) are standard PL 4-spheres

  (4-dimensional smooth Poincaré conjecture),

- whether every balanced presentation of the trivial group can be transformed into a trivial presentation by a sequence of Nielsen transformations

  (Andrews–Curtis conjecture).

Further limitations

[Milnor, 1966]

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[Lewiner, Lopes, Tavares, 2003; Joswig, Pfetsch, 2006]

Computing optimal discrete Morse functions is NP-hard.

Collapsing the k-simplex

Collapsing the *k*-simplex

k	Rounds	Got stuck	Percentage
7	10^{10}	0	0.0%
8	10^{9}	12	0.0000012%
9	10^{8}	2	0.000002%
10	10^{7}	3	0.00003%
11	10^{7}	12	0.00012%
12	10^{6}	4	0.0004%
13	10^{6}	6	0.0006%
14	10^{5}	4	0.004%
15	10^{5}	8	0.008%
16	10^{4}	4	0.04%
17	10^{4}	10	0.10%
18	10^{3}	2	0.2%
19	10^{3}	6	0.6%
20	10^{3}	13	1.3%
21	10^{3}	62	6.2%
22	10^{3}	153	15%
. 23	10^{2}	35	35%
24	10^{2}	67	67%
25	$5 \cdot 10^{1}$	46	92%

[Joswig, Lofano, L., Tsuruga, 2022]

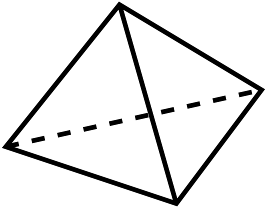
“The worst way to collapse a simplex”

[Lofano, Newman, 2019]

For $n \geq 8$ and $k \notin \{1, n-3, n-2, n-1\}$,

there is a collapsing sequence
of the $(n-1)$-simplex on n vertices
that gets stuck at dimension k.

This result is best possible.



Mousetraps I

[Adiprasito, Benedetti, L, 2017]

There is a contractible, but non-collapsible 3-dimensional simplicial complex `two_optima` with face vector $f = (106, 596, 1064, 573)$ that has two distinct optimal discrete Morse vectors, (1, 1, 1, 0) and (1, 0, 1, 1).

Mousetraps I

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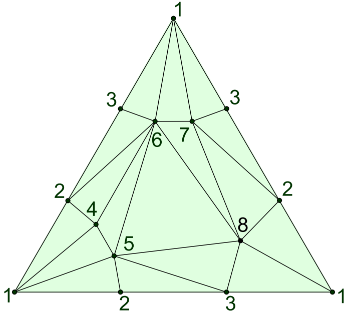
[Lofano, 2021+]

There is an 8-point Delaunay triangulation in $\mathbb{R}^3$ that collapses to a triangulation of the dunce hat with eight vertices. This example is smallest possible with respect to its number of vertices.

(This answers a question of Edelsbrunner.)

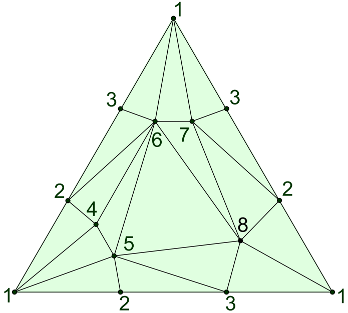
Mousetraps II



[Lofano, 2021+]

There is a simplicial complex with optimal discrete Morse vector $(1, 0, 0, 3)$ and whose best discrete Morse vector that can be found using random discrete morse is $(1, 1, 1, 3)$.

Mousetraps II



[Lofano, 2021+]

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The addition of the tetrahedron 1367 makes the triangulation collapsible.

On top of each of the triangles 136, 137, 167 boundaries of 4-simplices are added to block the tree triangles.

Horizon for computations

[Joswig, Lofano, L., Tsuruga, 2022]

Random collapses fail in dimensions $d \gg 25$.

Horizon for computations

[Joswig, Lofano, L., Tsuruga, 2022]

Random collapses fail in dimensions $d \gg 25$.

[Adiprasito, Benedetti, L, 2017]

Let K be any simplicial complex of dimension $d \geq 3$.
Then the random discrete Morse algorithm, applied to the k-th barycentric subdivision $\mathrm{sd}^k K$, yields an expected number of $\Omega(e^k)$ critical cells a.a.s.

Thank you!