Torsion Burst and Hadamard Matrix Torsion

Frank H. Lutz

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[Silesaurus, Krasiejów]

Random Models

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$G(n, p)$, n vertices, each edge with probability p

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Stochastic Process [Kahle, L., Newman, Parsons, 2018]

$\mathcal{Y}_d(n)$, n vertices, full $(d-1)$-skeleton,
d-faces one by one

The torsion burst for an instance of $\mathcal{Y}_2(75)$

[Kahle, L., Newman, Parsons, 2018]

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The torsion burst for an instance of $\mathcal{Y}_3(25)$

3-faces	H_3	H_2
1949	$\mathbb{Z}^4$	$\mathbb{Z}^{79}$
1950	$\mathbb{Z}^4$	$\mathbb{Z}^{78} \times \mathbb{Z}/6\mathbb{Z}$
1951	$\mathbb{Z}^4$	$\mathbb{Z}^{77} \times \mathbb{Z}/778016791830702358378590352176 0\mathbb{Z}$
1952	$\mathbb{Z}^5$	$\mathbb{Z}^{77} \times \mathbb{Z}/5\mathbb{Z}$
1953	$\mathbb{Z}^6$	$\mathbb{Z}^{77}$

The torsion burst for an instance of $\mathcal{Y}_4(17)$

4-faces	H_4	H_3
1787	$\mathbb{Z}^{10}$	$\mathbb{Z}^{43}$
1788	$\mathbb{Z}^{10}$	$\mathbb{Z}^{42} \times \mathbb{Z}/2\mathbb{Z}$
1789	$\mathbb{Z}^{10}$	$\mathbb{Z}^{41} \times \mathbb{Z}/2\mathbb{Z}$
1790	$\mathbb{Z}^{10}$	$\mathbb{Z}^{40} \times \mathbb{Z}/2\mathbb{Z}$
1791	$\mathbb{Z}^{10}$	$\mathbb{Z}^{39} \times \mathbb{Z}/4923498678446918889877 4\mathbb{Z}$
1792	$\mathbb{Z}^{11}$	$\mathbb{Z}^{39}$

The torsion burst for an instance of $\mathcal{Y}_5(16)$

5-faces	H_5	H_4
2972	$\mathbb{Z}^6$	$\mathbb{Z}^{37}$
2973	$\mathbb{Z}^6$	$\mathbb{Z}^{36} \times \mathbb{Z}/1147712621067945810235354141226409657574376675\mathbb{Z}$
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Conjecture (Łuczak and Peled, 2018)

For every $d \geq 2$ and $p = p(n)$ there is a constant c_d such that if $|np - c_d|$ is bounded away from 0, then $H_{d-1}(Y_d(n,p);\mathbb{Z})$ is torsion-free with high probability.

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[Aronshtam, Linial, 2013]: Torsion occurs near $p = c_d/n$, with $c_2 \approx 2.75381$, $c_3 \approx 3.90708$.

How much torsion can there be?

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[Kalai, 1983]

The highest possible asymptotic torsion growth
for 2-dimensional simplicial complexes with n vertices
is in $\Theta(2^{n^2})$.

Definition (Kalai, 1983)

A d-dimensional $\mathbb{Q}$-acyclic complex is

- *a d-dimensional simplicial complex X*
- *with complete $(d-1)$-skeleton,*
- *$\binom{n-1}{d}$ d-dimensional faces,*
- *$\beta_d(X,\mathbb{Q}) = 0$, and $\beta_{d-1}(X,\mathbb{Q}) = 0$.*

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$\mathbb{Q}$-acyclic *d*-complexes are higher-dimensional generalizations of trees. However, unlike trees, $\mathbb{Q}$-acyclic *d*-complexes may have finite but nontrivial $(d-1)$st homology.

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- $\mathbb{E}\left[|H_{d-1}(X)|\right] \geq \exp\left(\Theta\left(n^d\right)\right)$.
- $|H_{d-1}(X)| \leq \sqrt{d+1}^{\binom{n-2}{d}}$.

How to obtain explicit examples with high torsion?

Definition (Matrix Disc Complexes; Lofano & L, 2021)

Let $M = (M_{ij})$ be an $(m \times n)$-matrix with integer entries.

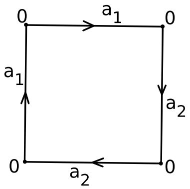
Matrix disc complexes $DC(M)$ associated with M comprise the 2-dimensional CW complexes constructed level-wise:

- Every complex in $DC(M)$ has a single 0-cell.
- The 1-skeleton of a complex in $DC(M)$ has an edge cycle a_j for every column index $j \in \{1, \ldots, n\}$ of the matrix M.
- Every row i of M with row sum $s_i = |M_{i1}| + \ldots + |M_{in}|$, $i \in \{1, \ldots, m\}$, contributes a polygonal disc with s_i edges.

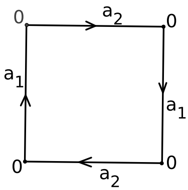
  For every positive entry M_{ij}, M_{ij} edges of the disc are oriented coherently and are assigned with the label a_j.

  In the case of a negative entry, the direction of the corresponding edges is reversed; in the case of a zero-entry, the respective edge does not occur.

$M = (2\ \ 2)$



Klein bottle



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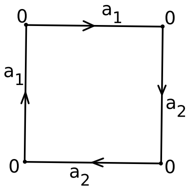

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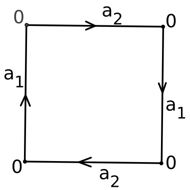
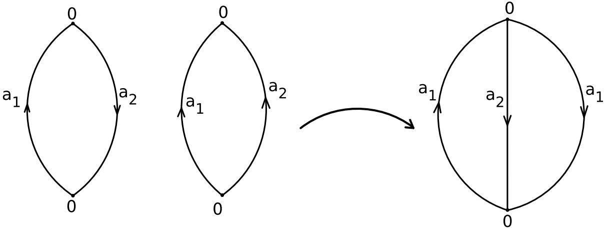


pinched $\mathbb{R}P^2$

$$H(2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



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- By construction, a representative $C \in DC(M)$ has a single vertex only, so C is connected and $H_0(C) = \mathbb{Z}$.

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- Each edge of C is a cycle, i.e., the first homology group $H_1(C)$ of C is determined by the n rows of M as relations, and therefore H_1 coincides for all the examples in $DC(M)$.

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- Further, the second homology H_2 of any representative C is simply the kernel of the matrix M.

In particular, if M is a square matrix with $det(M) \neq 0$, then $|H_1(C)| = |det(M)|$.

[Speyer, 2010]

There are matrix disc complexes that have triangulations with $\Theta(n)$ vertices and torsion growth $\Theta(2^n)$.

Speyer's construction

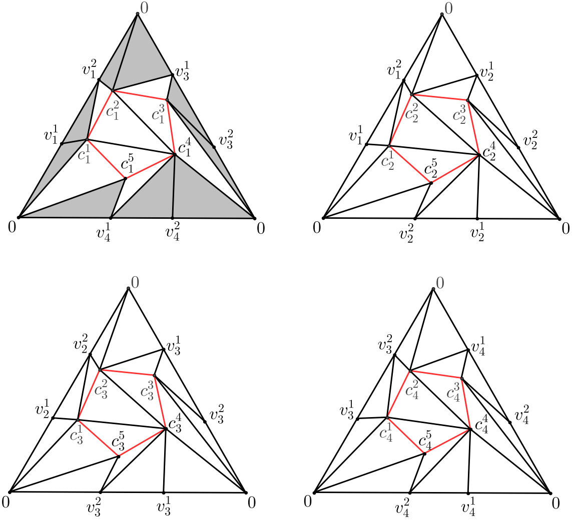
Let $k \geq 2$ be an integer and $k = \gamma_m 2^m + \gamma_{m-1} 2^{m-1} + \ldots + \gamma_0 2^0$ be its binary expansion, with leading coefficient $\gamma_m = 1$ and otherwise $\gamma_i \in \{0, 1\}$ for all $0 \leq i \leq m - 1$.

$M(k)$ is the $((m+1) \times (m+1))$-matrix:

- First row contains the entries $(-1)^i \gamma_{m-i}$ for $i \in \{0, \ldots, m\}$.
- The lower part of $M(k)$ has 1's on the first diagonal followed by 2's on the diagonal to the right, and all other entries equal to zero. It is then easy to see that $\det(M(k)) = k$.

For $k = 11 = 8 + 2 + 1$:

$$M(11) = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$



Four subdivided triangles of the complex associated to $M(11)$.

Hadamard matrix torsion

Hadamard matrices: $H(1) = (1)$

$$H(2^k) = \begin{pmatrix} H(2^{k-1}) & H(2^{k-1}) \\ H(2^{k-1}) & -H(2^{k-1}) \end{pmatrix}, \text{ for } k \geq 1$$

with $|\det(H(n))| = n^{n/2}$, for $n = 2^k$.

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[Lofano, L., 2021]

For each $n = 2^k,\ k \geq 1$, there is a $\mathbb{Q}$-acyclic 2-dimensional simplicial complex $\mathrm{HMT}(n)$ with

$$\begin{aligned} f(\mathrm{HMT}(n)) &= (5n-1, 3n^2+9n-6, 3n^2+4n-4), \\ H_*(\mathrm{HMT}(n)) &= (\mathbb{Z}, T(\mathrm{HMT}(n)), 0), \\ T(\mathrm{HMT}(n)) &= (\mathbb{Z}_2)^{\binom{k}{1}} \times (\mathbb{Z}_4)^{\binom{k}{2}} \times \cdots \times (\mathbb{Z}_{2^k})^{\binom{k}{k}}, \\ |T(\mathrm{HMT}(n))| &= n^{n/2} \in \Theta(2^{n \log n}). \end{aligned}$$

The examples $\mathrm{HMT}(n)$ can be constructed algorithmically in quadratic time $\Theta(n^2)$.