Slices of the Takagi function and (other) self-affine sets

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joint with B. Bárány and A. Käenmäki

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Thermodynamic Formalism: Non-additive Aspects and Related Topics, Będlewo

Takagi function

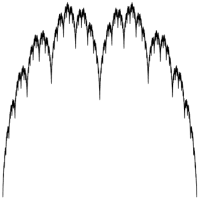

Figure: The graph of the Takagi function for $\lambda = 2/3$.

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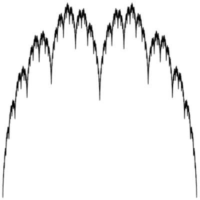

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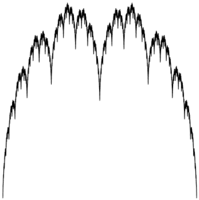

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- A well known example of a "pathological" continuous but nowhere differentiable function.

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- What can we say about all slices?

Main result

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Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, then

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_{\mathrm{H}}(T_\lambda \cap (V + x)) = \dim_{\mathrm{A}}(T_\lambda) - 1 < 1.$$

Weak tangents and Assouad dimension

Let $X \subset \mathbb{R}^2$ be compact and $T_{x,r}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity taking $Q(x,r) := x + [0,r]^2$ to the unit cube $Q = [0,1]^2$ in an orientation preserving way.

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$$T_{x_n,r_n}(X) \cap Q \to T$$

in the Hausdorff distance, then T is called a weak tangent of X.

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Theorem (Käenmäki-Ojala-Rossi, 2018)

If $X \subset \mathbb{R}^2$ is a compact set, then

$$\dim_{\mathrm{A}}(X) = \max\{\dim_{\mathrm{H}}(T) \colon T \in \mathrm{Tan}(X)\}.$$

Self-affine sets

A finite collection $\{\varphi_i(x) = A_i x + t_i\}_{i=1}^{M}$ of invertible contractive affine maps on $\mathbb{R}^2$ is called a self-affine iterated function system (affine IFS).

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We say that X satisfies the strong separation condition if $\varphi_i(X) \cap \varphi_j(X) = \emptyset$, for all $i \neq j$, The associated symbolic space of infinite words is denoted by $\Sigma = \{1, \ldots, M\}^{\mathbb{N}}$ and the set of finite words of length n by Σ_n. The elements of these spaces are denoted by $\mathtt{i} := (i_1, i_2, \ldots)$.

First example

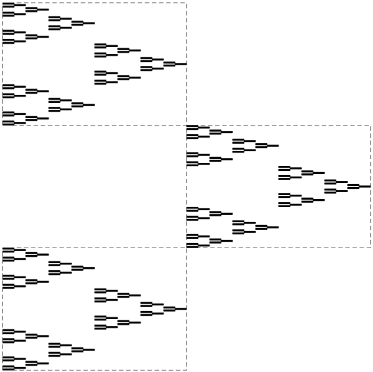


Figure: A Bedford-McMullen carpet

Second example

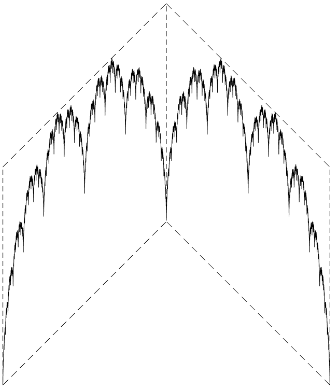
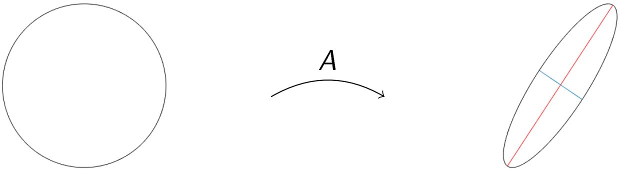


Figure: The Takagi function is an attractor of an affine IFS

Definitions



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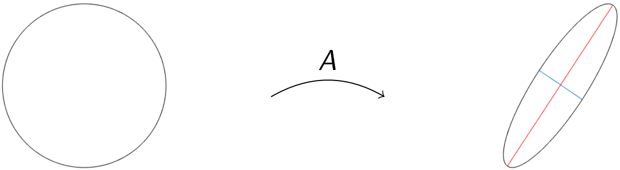

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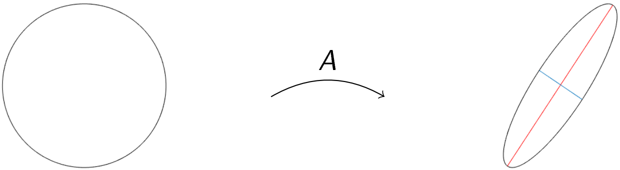

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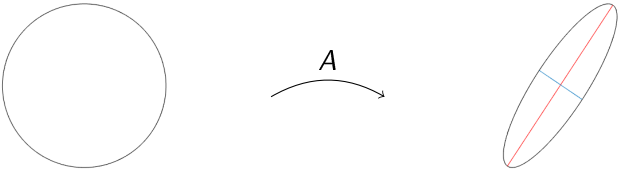

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- Let $\vartheta(A)$ denote the line spanned by the longer semiaxis of $A(B(0,1))$.

Definitions

- For $\mathtt{i} \in \Sigma$, let

$$\overline{\vartheta}_1(\mathtt{i}) = \lim_{n\to\infty} \vartheta(A_{i_1} \cdot \ldots \cdot A_{i_n}),$$
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  if the limit exists.
- Geometric interpretation: $\overline{\vartheta}_1(\mathtt{i})$ are the limiting directions of the construction cylinders.

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- A self-affine set X is dominated if there exist constants $C > 0$ and $0 < \tau < 1$, such that

$$\frac{\alpha_2(A_{i_1} \cdot \ldots \cdot A_{i_n})}{\alpha_1(A_{i_1} \cdot \ldots \cdot A_{i_n})} \leqslant C\tau^n,$$

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Lemma

If X is dominated, then the limit directions $\overline{\vartheta}_1(\mathtt{i})$ and $\overline{\vartheta}_2(\mathtt{i})$ exist for all $\mathtt{i} \in \Sigma$ and the convergence is uniform. Moreover, the sets $Y_F := \overline{\vartheta}_1(\Sigma)$ and $X_F := \overline{\vartheta}_2(\Sigma)$ are disjoint compact sets.

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- We call the sets Y_F and X_F the forward and backward Furstenberg directions, respectively.

Assouad dimension of self-affine sets

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Theorem (Bárány-Käenmäki-Yu, 2023)

Let X be a strongly separated, dominated self-affine set with $\dim_{\mathrm{H}}(X) \geqslant 1$ and such that X_F is not a singleton. Then

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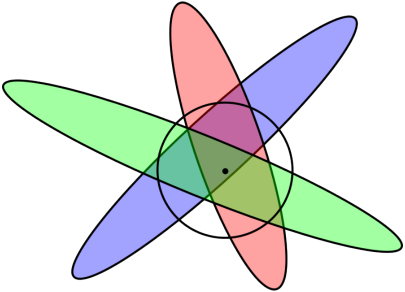
- We want to apply this to the Takagi function.
- Strong separation condition is not satisfied.

Bounded neighbourhood condition

A self-affine set X satisfies the bounded neighbourhood condition (BNC) if there is a constant M, such that

$$\#\{\varphi_{\mathtt{i}} \mid \alpha_2(A_{\mathtt{i}}) \approx r,\ B(x,r) \cap \varphi_{\mathtt{i}}(X) \neq \emptyset\} \leqslant M,$$

for all $x \in X$ and $r > 0$.



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Theorem (A.-Bárány-Käenmäki, 2023)

If X is a dominated self-affine set satisfying the BNC, such that $\dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp} X) = 1$ for all $V \in X_F$, then

$$\dim_{\mathrm{A}}(X) = 1 + \max_{\substack{x \in X \\ V \in X_F}} \dim_{\mathrm{H}}(X \cap (V + x))$$
$$= 1 + \max_{\substack{x \in X \\ V \in \mathbb{RP}^1 \setminus Y_F}} \dim_{\mathrm{A}}(X \cap (V + x)).$$

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- The Takagi function is a dominated self-affine set and satisfies the BNC.
- Since T_λ is continuous, it projects to a line segment in all directions and in particular $\dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp} T_\lambda) = 1$, for all $V \in \mathbb{RP}^1$.

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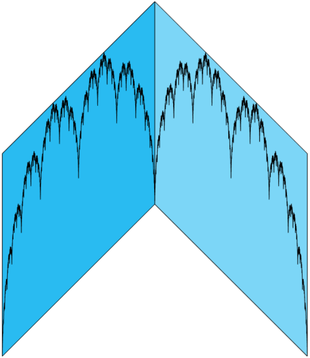

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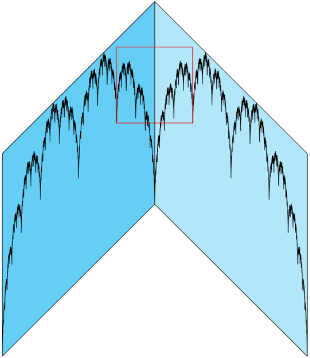

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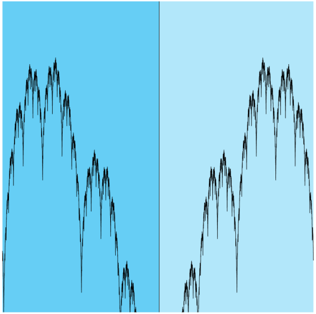

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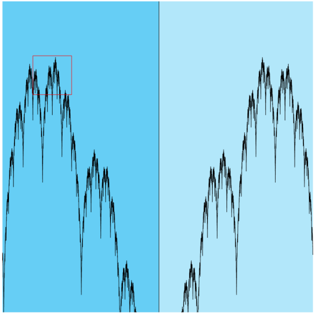

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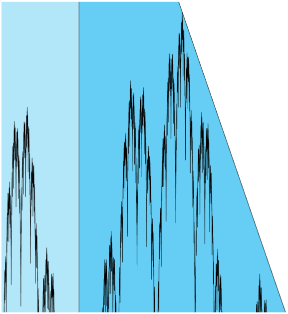

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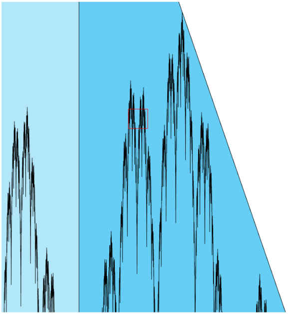

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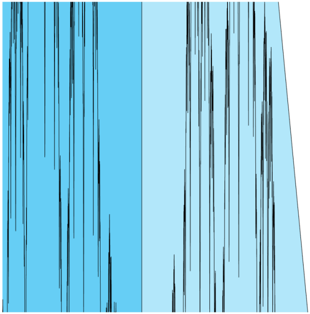

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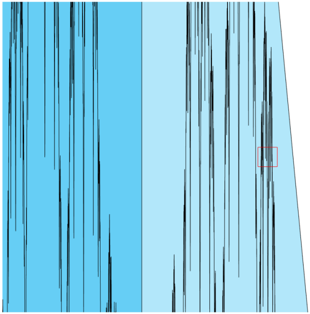

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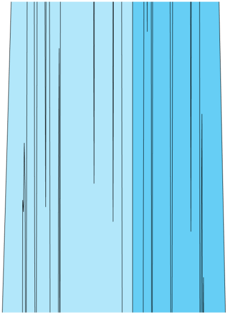

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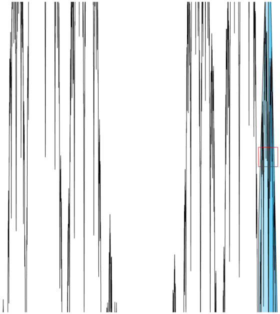

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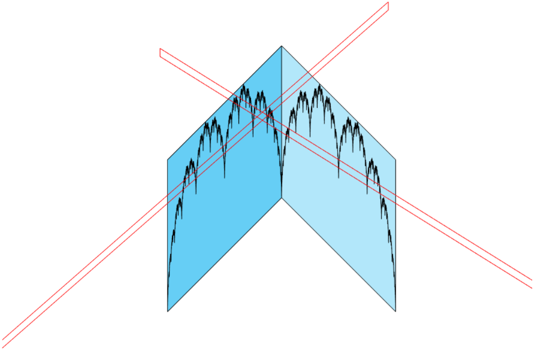

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- $\dim_{\mathrm{H}}(T) \leqslant 1 + \dim_{\mathrm{H}}(X \cap (V + x))$, where $V \in X_F$.
- $\dim_{\mathrm{A}}(X) \leqslant 1 + \max_{\substack{x \in X \\ V \in X_F}} \dim_{\mathrm{H}}(X \cap (V + x))$.

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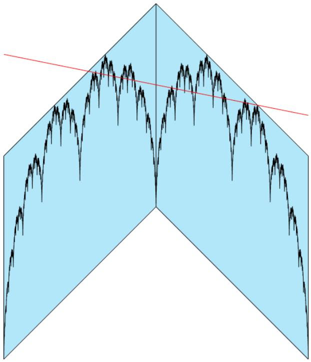

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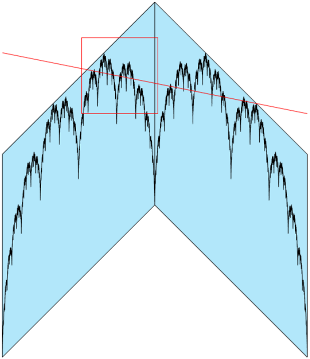

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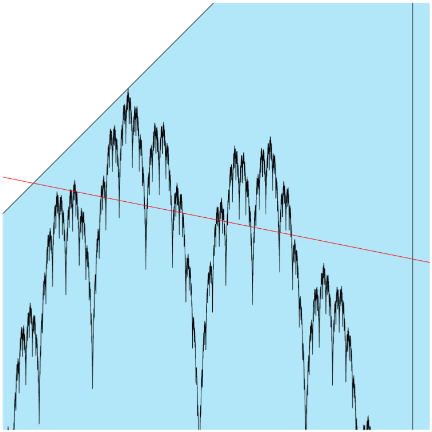

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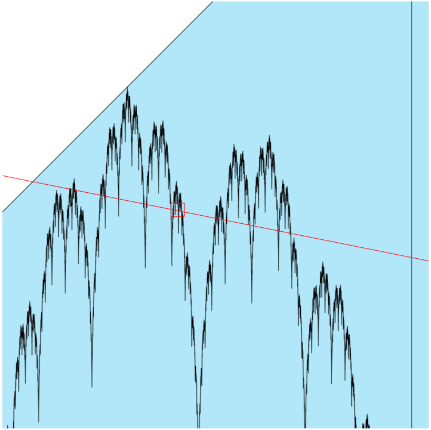

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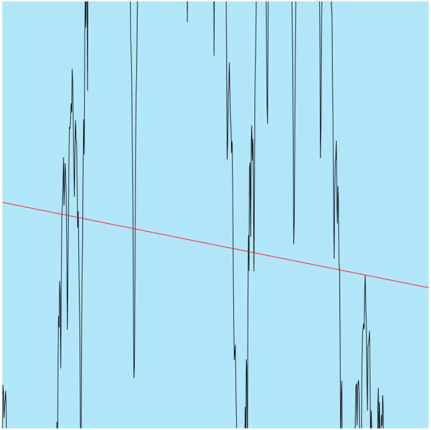

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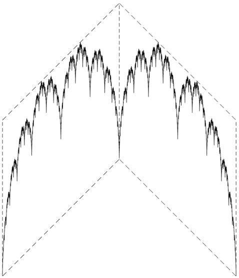
- $\dim_{\mathrm{A}}(X) \geqslant \dim_{\mathrm{H}}(T) \geqslant 1 + \dim_{\mathrm{H}}(S) = 1 + \dim_{\mathrm{A}}(X \cap (V + x)) \geqslant 1 + \dim_{\mathrm{H}}(X \cap (V + x))$.

□

Back to the Takagi function



Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, then

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_H(T_\lambda \cap (V + x)) = \dim_A(T_\lambda) - 1 < 1.$$

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  1. $\dim_A(T_\lambda) = \dim_H(T_\lambda)$, and the bound of Marstrand's slicing theorem extends to all slices.
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Extending Marstrand's theorem

Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, and μ is the projection of the uniform Bernoulli measure on the symbolic space to T_λ, then $\dim_A(T_\lambda) = \dim_H(T_\lambda)$ and in particular

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_H(T_\lambda \cap (V + x)) = \dim_H(T_\lambda) - 1,$$

if and only if

$$\overline{\dim}_{loc}(\mathrm{proj}_{V^\perp *}\mu, \mathrm{proj}_{V^\perp}(x)) \geqslant 1,$$

for all $x \in T_\lambda$ and $V \in X_F$.

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- The proof is a technical geometric argument, which establishes a connection between the local dimensions of the projected measure and the box dimensions of the slices of the set along the fibers.
- Unfortunately, we do not know any values of λ where either of the conditions hold.

Thank you for your attention!

Questions are welcome!