Sparse sampling and dilation operations on a Gibbs weighted tree, and multifractal formalism

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Thermodynamic Formalism: Non-additive Aspects and Related Topics,

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Foreword: A model of multifractal lacunary wavelet series

Denote by $\mathcal{D}$ the set of dyadic subintervals of $[0,1]$, and for all $j \geq 0$, $\mathcal{D}$ the set of dyadic subintervals of $[0,1]$ of generation j

Consider a Meyer wavelet ψ so that the family $(\psi_I)_{I\mathcal{D}}$ is orthogonal, with

$$\psi_I(x) = \psi(2^j x - k) \quad \text{if } I = [k2^{-j}, (k+1)2^{-j}].$$

Consider $\gamma > 0$ and a simple model of γ-Hölder monofractal function over $[0,1]$:

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Then, S. Jaffard (2000) essentially considers the following model of sparse wavelet series: introduce a **lacunarity parameter** $\eta \in (0,1)$, $(p_I)_{I\in\mathcal{D}}$ a sequence of independent Bernoulli variables, such that $p_I \sim B(2^{-(1-\eta)j})$ if $I \in \mathcal{D}_j$, and consider

$$\sum_{I\in\mathcal{D}} p_I \, |I|^\gamma \psi_I, \quad \text{where } \#\{I \in \mathcal{D}_j : p_I = 1\} \approx 2^{j\eta}.$$

This new series is multifractal.

Foreword: A model of multifractal lacunary wavelet series

Recall that if $f : [0,1] \mapsto \mathbb{R}$ is bounded, for all $x_0 \in [0,1]$, the pointwise Hölder exponent of f is defined as

$$h_f(x_0) = \sup\{h \geq 0 : \exists\, P \in \mathbb{R}[X],\ \deg(P) \leq \lfloor h \rfloor,\ f(x) - P(x - x_0) = O(|x - x_0|^h)\}$$

Then, the multifractal spectrum of f is defined as

$$\sigma_f : H \in \mathbb{R} \cup \{\infty\} \mapsto \dim\{x \in [0,1] :\ h_f(x) = H\}.$$

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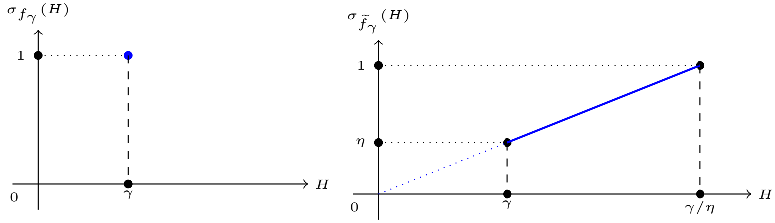
For $f_\gamma = \sum_{I \in \mathcal{D}} |I|^\gamma \psi_I$ and $\widetilde{f}_\gamma = \sum_{I \in \mathcal{D}} p_I\, |I|^\gamma \psi_I$, one has



Explanation. Properties of the intervals which survive.

For $j \geq 0$ and $I = [k2^{-j}, (k+1)2^{-j}] \in \mathcal{D}_j$, let $x_I = k2^{-j}$.

Let

$$\mathcal{S}_j = \{I \in \mathcal{D}_j : p_I = 1\}.$$

Proposition

There exists a decreasing sequence $(\varepsilon_j)_{j\geq 0}$ *converging to* 0 *such that, with probability 1, for* j *large enough,*

$$\forall\, J \in \mathcal{D}_{\lfloor j(\eta-\varepsilon_j)\rfloor},\ \#1 \leq \{I \in \mathcal{S}_j : I \subset J\} \leq 2^{j\varepsilon_j}.$$

Definition

For $x \in [0,1]$, the rate of approximation of x by $\{(x_I, r_I = 2^{-j(\eta-\varepsilon_j)})\}_{j\geq 0,\, I\in\mathcal{S}_j}$

$$\delta_x = \limsup_{j\to\infty} \frac{\log d(x, \{x_I : I \in \mathcal{S}_j\})}{\log 2^{-j\eta}}.$$

Note that $\delta_x \geq 1$ for all $x \in [0,1]$.

Explanation. Wavelet characterization of the pointwise exponent, and ubiquity properties.

(1) Due to Jaffard, if $f=\sum_{I\in\mathcal{D}} c_I\psi_I$ is Hölder continuous, then

$$h_f(x)=\liminf_{j\to\infty}\frac{\log\sup\{|c_I| : I\subset 3I_j(x)\}}{\log 2^{-j}}.$$

One can then check that

$$\forall x\in[0,1],\ h_{\widetilde{f}_\gamma}(x)=\liminf_{j\to\infty}\frac{\log\sup\{p_I|I|^\gamma\ :\ I\subset 3I_j(x)\}}{\log 2^{-j}}=\frac{\gamma}{\min(\eta^{-1},\delta_x)}.$$

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(2) One deduces from the proposition that

$$\mathrm{Leb}\Big(\bigcap_{k\ge 0}\bigcup_{j\ge k}\bigcup_{I\in\mathcal{S}_j} B\big(x_I,2^{-j(\eta-\varepsilon_j)}\big)\Big)=1,$$

and

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Then, by the mass transference principle (Jaffard ('00), Beresnevich-Velani ('08))

$$\forall\,\delta \ge 1,\ \dim\{x\in[0,1] : \delta_x = \delta\} = \dim\{x\in[0,1] : \delta_x \ge \delta\} = \frac{1}{\delta},$$

$$\text{hence } \sigma_{\widetilde{f}_\gamma}(H) = \begin{cases} \frac{\eta}{\gamma} H & \text{if } H \in [\gamma, \gamma/\eta] \\ -\infty & \text{otherwise} \end{cases}.$$

Capacities

Now, one can more generally consider any Hölder continuous wavelet series

$$f = \sum_{I \in \mathcal{D}} c_I \psi_I \text{ and make it lacunary } \widetilde{f} = \sum_{I \in \mathcal{D}} p_I \, c_I \psi_I.$$

Recall that the multifractal analysis of these functions consists respectively in the multifractal analysis of the level sets of the exponents

$$h_f(x) = \liminf_{j \to \infty} \frac{\log \sup\{|c_I| : I \subset 3I_j(x)\}}{\log 2^{-j}}$$

$$h_{\widetilde{f}}(x) = \liminf_{j \to \infty} \frac{\log \sup\{p_I c_I : I \subset 3I_j(x)\}}{\log 2^{-j}}.$$

This reduces back to the multifractal analysis of some non-decreasing functions of intervals, namely capacities.

Precisely, given a real sequence $(c = c_I)_{I \in \mathcal{D}}$, define, for any interval $J \subset [0,1]$

$$\mathsf{M}(c)(J) = \sup\{|c_I| : I \in \mathcal{D}, \, I \subset J\}.$$

Clearly, $J \subset K \Longrightarrow \mathsf{M}(c)(J) \leq \mathsf{M}(c)(K)$ and $h_f(x) = \liminf_{r \to 0^+} \dfrac{\log \left(\mathsf{M}(c)(B(x,r)) \right)}{\log(r)}$.

Multifractal analysis of capacities on $[0,1]$. Multifractal spectrum

If μ is a capacity over $[0,1]$, its topological support is defined as

$$\text{supp}(\mu) = \{x \in [0,1] : \mu(B(x,r)) > 0, \ \forall r > 0\}.$$

We will assume that $\text{supp}(\mu) = [0,1]$ and define the L^q-spectrum of μ as

$$\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{j\to\infty} \underbrace{-\frac{1}{j} \log_2 \sum_{I \in \mathcal{D}_j} \mu(I)^q}_{\tau_{\mu,j}(q)}$$

The *multifractal spectrum* of μ is then defined as

$$\sigma_\mu : H \mapsto \dim \underline{E}_\mu(H), \quad H \in \mathbb{R},$$

where

$$\underline{E}_\mu(H) = \left\{ x \in \text{supp}(\mu) : \liminf_{j\to\infty} \frac{\log\left(\mu(3I_j(x))\right)}{\log(|3I_j(x)|)} = H \right\}.$$

One has

$$\sigma_\mu(H) \leq \tau_\mu^*(H) = \inf\{Hq - \tau_\mu(q) : q \in \mathbb{R}\} \in \mathbb{R}_+ \cup \{-\infty\}.$$

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Multifractal analysis of capacities on $[0,1]$. Large deviations spectra

One is also interested also in the asymptotic statistical distribution of μ via large deviations:

(1) Do we have

$$\tau_\mu(q) = \lim_{j\to\infty} \tau_{\mu,j} \text{ ?}$$

(2) For $H \in \mathbb{R}$, $j \in \mathbb{N}$ and $\varepsilon > 0$, let

$$\mathcal{E}_\mu(j, H\pm\varepsilon) = \left\{ I \in \mathcal{D}_j : \frac{\log \mu(I)}{\log(|I|)} \in [H-\varepsilon, H+\varepsilon] \right\}.$$

Then, the lower and upper large deviations spectra of μ are respectively defined as

$$\underline{f}_\mu(H) = \lim_{\varepsilon\to 0} \liminf_{j\to+\infty} \frac{\log_2 \#\mathcal{E}_\mu(j, H\pm\varepsilon)}{j}$$

$$\text{and} \quad \overline{f}_\mu(H) = \lim_{\varepsilon\to 0} \limsup_{j\to+\infty} \frac{\log_2 \#\mathcal{E}_\mu(j, H\pm\varepsilon)}{j}.$$

Does the following large deviations principle hold:

$$\underline{f}_\mu(H) = \overline{f}_\mu(H) = \tau_\mu^* \text{ ?}$$

The fundamental example of Gibbs measures and capacities

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be $\mathbb{Z}$-invariant Hölder continuous potential. It is standard that

$$\nu_n(\mathrm{d}x) = \frac{\exp\left(S_n\varphi(x)\right)}{\int_{[0,1]} \exp\left(S_n\varphi(t)\right) \mathrm{Leb}(\mathrm{d}t)} \mathrm{Leb}(\mathrm{d}x), \quad \text{where } S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(2^n x),$$

converges vaguely to a measure fully supported on $\mathbb{R}$ (Ruelle). Denote by ν the restriction of this *Gibbs* measure to $[0,1]$.

Definition

A Gibbs capacity μ is a capacity such that $\mu \approx \nu^\gamma$, where $\gamma > 0$ and ν is a Gibbs measure on $[0,1]$ as above.

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Classical results on Gibbs measures (Ruelle,Collet-Lebowitz-Porzio) give :

- $\tau_\nu = \lim_{j\to\infty} \tau_{\nu,j}$. Moreover, τ_ν is analytic, concave increasing and $\lim_{q\to+\infty} \tau_\nu(q) = +\infty$.
- $\sigma_\nu = \tau_\nu^*$; also $\mathrm{dom}(\sigma_\mu) = [\tau'_\mu(\infty, \tau'_\mu(-\infty)] \subset (0,\infty)$.
- The large deviations principle $\underline{f}_\mu(H) = \overline{f}_\mu(H) = \tau_\mu^*$ holds.

The same properties are true for any Gibbs capacity μ.
Moreover, μ is multifractal, i.e. $\mathrm{dom}(\sigma_\mu)$ is not a singleton, iff φ is not cohomologous to a constant on $\mathbb{R}/\mathbb{Z}$ endowed with the dynamics $\times 2$, i.e. $\mu \not\approx \mathrm{Leb}^\gamma$.

Dilation operation on capacities

Let μ be a fully supported capacity on $[0,1]$. For any $\varrho \in (0,1)$, we define the dilation operation on μ restricted to $\mathcal{D}$:

$$c_\varrho(\mu)(I) = \mu(I^\varrho) \quad \text{(should be thought of as wavelet coefficients)}$$

where, if $I \in \mathcal{D}_j$, I^ϱ is the dyadic interval of generation $\lfloor \varrho j \rfloor$ that contains I.

Then we consider the capacity

$$\mathsf{M}_\varrho(\mu) = \mathsf{M}(c_\varrho(\mu)) : J \mapsto \sup\{\mu(I^\varrho) : I \in \mathcal{D},\, I \subset J\}.$$

Observe that if M is doubling, then

$$\mathsf{M}_\varrho(B(x,r)) \approx \mu(B(x,r^\varrho));$$

also

$$\sigma_{\mathsf{M}_\varrho(\mu)}(H) = \sigma_\mu(H/\varrho) \text{ for all } H \in \mathbb{R}$$
$$\tau_{\mathsf{M}_\varrho(\mu)}(q) = \varrho - 1 + \varrho\tau_\mu(q) \text{ for all } q \in \mathbb{R}.$$

In particular, $\tau^*_{\mathsf{M}_\varrho(\mu)}(H) = \varrho\tau^*_\mu(H/\varrho) + 1 - \varrho$, so if μ is multifractal and $\varrho \in (0,1)$, then the multifractal formalism fails everywhere for M_ϱ, except at $H = \varrho\tau_\mu(0)$.

Combining dilation with sparse sampling (or lacunarisation) operation

Remark: In the very beginning, we considered the situation $\mu = \mathrm{Leb}^\gamma$. In this case, $\mu(I^\varrho) \approx \mathrm{Leb}^{\gamma\varrho}$, so we stay in the class of monofractal Gibbs capacities.

Finally, we consider, for $\varrho \in (0, 1/\eta]$,

$$\widetilde{c}_{\varrho\eta}(\mu)(I) = p_I \mu(I^{\varrho\eta})$$

(should be thought of as wavelet coefficients of a lacunary series), and

$$\mathsf{M}_{\varrho,\eta} = \mathsf{M}(\widetilde{c}_{\varrho\eta}(\mu)) : J \mapsto \sup\{p_I \mu(I^{\varrho\eta}) : I \in \mathcal{D}, I \subset J\}.$$

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Goals: (1) Discuss the validity of the multifractal formalism and the large deviations principle for $\mathsf{M}_{\varrho,\eta}$;
(2) Express $\sigma_{\mathsf{M}_{\varrho,\eta}}$ and $\tau_{\mathsf{M}_{\varrho,\eta}}$ in terms of σ_μ and τ_μ respectively, and discuss the real analyticity properties of these spectra.

Expectation: That it be feasible, and the model be versatile.

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From now we take μ in the set of multifractal Gibbs capacities.

Validity of the LDP and MF for $\mathsf{M}_{\varrho,\eta}$

Fix μ a multifractal Gibbs capacity. For $\varrho \in (0, 1/\eta]$, let

$$\mathsf{M}_{\varrho,\eta} = \mathsf{M}(\widetilde{c}_{\varrho\eta}(\mu)) : J \mapsto \sup\{p_I \mu(I^{\varrho\eta}) : I \in \mathcal{D}, I \subset J\}.$$

Let us start with a qualitative rather than very explicit result.

Theorem (LDP and MF for $\mathsf{M}_{\varrho,\eta}$)

Fix $\varrho \in (0, 1/\eta]$. With probability 1,

1. ***(LDP)*** $\tau_{\mathsf{M}_{\varrho,\eta}} = \lim_{j\to\infty} \tau_{\mathsf{M}_{\varrho,\eta},j}$, *and the LDP holds.*

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   - *If $\varrho \in [1, 1/\eta]$, then the MF holds for $\mathsf{M}_{\varrho,\eta}$ over $\mathrm{dom}(\sigma_{\mathsf{M}_{\varrho,\eta}})$.*
   - *If $\varrho \in [0, 1)$, then $\mathrm{dom}(\sigma_{\mathsf{M}_{\varrho,\eta}}) = [\varrho\eta\tau'_\mu(\infty), \varrho\tau'_\mu(-\infty)]$, and there exists a strict compact subinterval $J_{\varrho,\eta}$ of $[\varrho\eta\tau'_\mu(\infty), \varrho\tau'_\mu(0))$ such that the MF holds only at points of $J_{\varrho,\eta} \cup \{\varrho\tau'_\mu(0)\}$.*

     Moreover, the three possible situations can occur, i.e. $J_{\varrho,\eta}$ can be either non trivial, a singleton or empty.

Formula for $\tau_{\mathsf{M}_{\varrho,\eta}}$ when $\varrho \in (0,1)$.

Theorem (L^q-spectrum of $\mathsf{M}_{\varrho,\eta}$; $\varrho \in (0,1]$)

Fix $\varrho \in (0,1]$. Let q_ϱ be the unique solution of $\varrho - 1 + \varrho\tau_\mu(q) = 0$. With probability 1,

$$\tau_{\mathsf{M}_{\varrho,\eta}(\mu)}(q) = \begin{cases} \varrho - 1 + \varrho\tau_\mu(q) & \text{if } q < q_\varrho, \\ \eta(\varrho - 1 + \varrho\tau_\mu(q)) & \text{if } q \geq q_\varrho. \end{cases}$$

In particular there is a first order phase transition at q_ϱ, and τ_μ is real analytic over $\mathbb{R} \setminus \{q_\varrho\}$.

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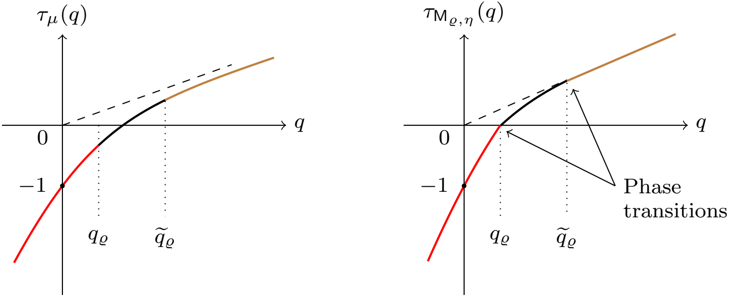
Let $\widetilde{H}_\varrho = \min\{H \geq 0 : \sigma_\mu(H) \geq 1 - 1/\varrho\}$ and $\widetilde{q}_\varrho = \sigma'_\mu(\widetilde{H}_\varrho)$.

With probability 1,

$$\tau_{\mathsf{M}_{\varrho,\eta}(\mu)}(q) = \begin{cases} \tau_\mu(q) - \frac{\tau_\mu(q_\varrho)}{q_\varrho} \cdot q & \text{if} \quad q \leq q_\varrho, \\ \eta(\varrho - 1 + \varrho\tau_\mu(q)) & \text{if} \quad q_\varrho < q < \widetilde{q}_\varrho, \\ \eta\varrho\tau'_\mu(\widetilde{q}_\varrho) \cdot q & \text{if} \quad \widetilde{q}_\varrho < +\infty \text{ and } q \geq \widetilde{q}_\varrho. \end{cases}$$

In particular there is a first order phase transition at q_ϱ, and possibly a second order phase transition at $\widetilde{q}_\varrho$ if this number is finite, i.e. if $\sigma_\mu(\tau'_\mu(\infty)) < 1 - 1/\varrho$.

Also, τ_μ is real analytic over $\mathbb{R} \setminus \{q_\varrho, \widetilde{q}_\varrho\}$.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Phase transitions

Formula for $\sigma_{\mathsf{M}_{\varrho,\eta}}$ when $\varrho \in [1, 1/\eta]$

Here $\sigma_{\mathsf{M}_{\varrho,\eta}} = \tau^*_{\mathsf{M}_{\varrho,\eta}}$.

Theorem (Multifractal spectrum of $\mathsf{M}_{\varrho,\eta}$ for $\varrho \in [1, 1/\eta]$)

Fix $\varrho \in [1, 1/\eta]$. Let q_ϱ be the unique solution of $\varrho - 1 + \varrho\tau_\mu(q) = 0$. Let $H_\varrho = \tau'_\mu(q_\varrho)$, $\widetilde{H}_\varrho = \min\{H \geq 0 : \sigma_\mu(H) \geq 1 - 1/\varrho\}$ and $\widehat{H}_\varrho = -\frac{\tau_\mu(q_\varrho)}{q_\varrho}$.

With probability 1, $\mathsf{M}_{\varrho,\eta}$ satisfies the multifractal formalism with

$$\sigma_{\mathsf{M}_{\varrho,\eta}(\mu)}(H) = \begin{cases} \eta(1 - \varrho + \varrho\sigma_\mu(H/\varrho\eta)) & \text{if } \varrho\eta\widetilde{H}_\varrho \leq H < \varrho\eta H_\varrho, \\ q_\varrho H & \text{if } \varrho\eta H_\varrho \leq H < H_\varrho + \widehat{H}_\varrho, \\ \sigma_\mu\big(H - \widehat{H}_\varrho\big) & \text{if } H_\varrho + \widehat{H}_\varrho \leq H \leq \tau'_\mu(\infty) + \widehat{H}_\varrho, \\ -\infty & \text{otherwise.} \end{cases}$$

Spectrum of $\mathsf{M}_{1,\eta}$

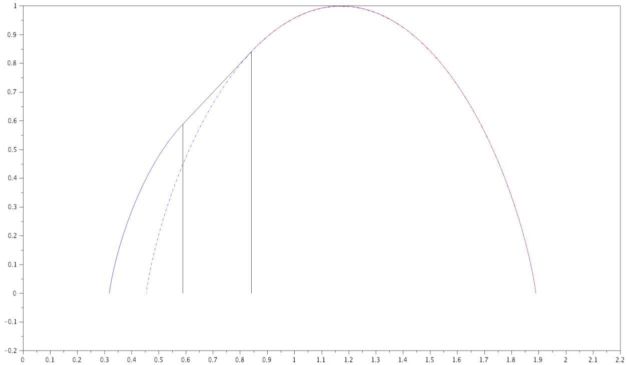

Figure: Spectrum of $\mathsf{M}_{1,\eta}$ for a given spectrum of μ, $\eta = 0.7$ here: 3 phases.

Formula for $\sigma_{\mathsf{M}_{\varrho,\eta}}$ when $\varrho \in (0,1)$

Here, $\sigma_{\mathsf{M}_{\varrho,\eta}} \neq \tau^*_{\mathsf{M}_{\varrho,\eta}}$.
For $H \geq 0$, set

$$D_{\varrho,\eta}(H) = \max\left\{\min\left(\frac{1-\varrho+\varrho\sigma_\mu(h)}{\delta}, \sigma_\mu(h)\right) : 0 \leq \frac{\varrho h}{\delta} \leq H,\ 1 \leq \delta \leq 1/\eta\right\}.$$

Theorem (Multifractal spectrum of $\mathsf{M}_{\varrho,\eta}$; $\varrho \in (0,1)$)

Fix $\varrho \in (0,1)$. With probability 1,

$$\sigma_{\mathsf{M}_{\varrho,\eta}}(H) = \begin{cases} D_{\varrho,\eta}(H) & \text{if } \eta\varrho\tau'_\mu(\infty) \leq H \leq \varrho\tau'_\mu(0) \\ \sigma_\mu(H/\varrho) & \text{if } \varrho\tau'_\mu(0) \leq H \leq \varrho\tau'_\mu(-\infty) \\ -\infty & \text{otherwise} \end{cases}$$

Explicit formula for $D_{\varrho,\eta}$; case $\sigma_\mu(\tau_\mu'(\infty)) > \frac{1-\varrho}{1/\eta-\varrho}$

Let θ_ϱ be the mapping defined as

$$\theta_\varrho : H \in [\tau_\mu'(\infty), \tau_\mu'(0)] \mapsto \frac{\varrho H \sigma_\mu(H)}{1-\varrho+\varrho\sigma_\mu(H)}.$$

Suppose that $\sigma_\mu(\tau_\mu'(\infty)) > \frac{1-\varrho}{1/\eta-\varrho}$. One has

$$D_{\mu,\varrho,\eta}(H) = \begin{cases} \eta(1-\varrho+\varrho\sigma_\mu(H/\varrho\eta)) & \text{if } \varrho\eta\tau_\mu'(\infty) \le H < \varrho\eta H_\varrho, \\ \dfrac{\sigma_\mu(H_\varrho)}{\theta_\varrho(H_\varrho)} H & \text{if } \varrho\eta H_\varrho \le H < \theta_\varrho(H_\varrho), \\ \sigma_\mu(\theta_\varrho^{-1}(H)) & \text{if } \theta_\varrho(H_\varrho) \le H < \varrho\tau_\mu'(0), \\ \sigma_\mu(H/\varrho) & \text{if } \varrho\tau_\mu'(0) \le H \le \varrho\tau_\mu'(-\infty), \end{cases} \quad (1)$$

and $D_{\mu,\varrho,\eta}(H) = -\infty$ otherwise.

Explicit formula for $D_{\varrho,\eta}$; case $\sigma_\mu(\tau'_\mu(\infty)) > \frac{1-\varrho}{1/\eta-\varrho}$

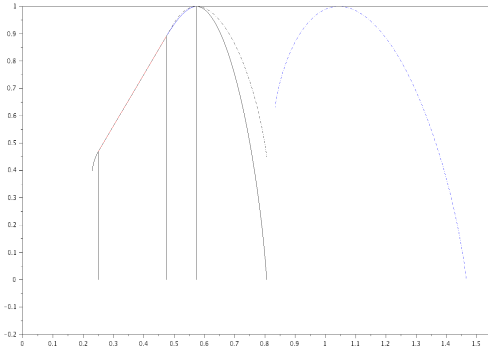

Figure: Spectrum of $\mathsf{M}_{\varrho,\eta}$ when $\varrho = 0.55$, $\eta = 0.5$, case $\sigma_\mu(\tau'_\mu(\infty)) > \frac{1-\varrho}{1/\eta-\varrho}$: 4 phases. Original spectrum in dashed blue, Legendre spectrum in dashed black.

Explicit formula for $D_{\varrho,\eta}$; case $\sigma_\mu(\tau'_\mu(\infty)) \leq \frac{1-\varrho}{1/\eta-\varrho}$

Suppose that $\sigma_\mu(\tau'_\mu(\infty)) \leq \frac{1-\varrho}{1/\eta-\varrho}$. Let $H_{\varrho,\eta} \in [H_{\min}, \tau'_\mu(0)]$ be the unique solution to

$$\sigma_\mu(H_{\varrho,\eta}) = \frac{1-\varrho}{1/\eta-\varrho}.$$

Let $H_\varrho = \tau'_\mu(q_\varrho)$, where $\varrho - 1 + \varrho\tau_\mu(q_\varrho) = 0$.

1. If $H_{\varrho,\eta} < H_\varrho$, then set

$$D_{\mu,\varrho,\eta}(H) = \begin{cases} \sigma_\mu(H/\varrho\eta) & \text{if } \varrho\eta\tau'_\mu(\infty) \leq H < \varrho\eta H_{\varrho,\eta}, \\ \eta(1-\varrho+\varrho\sigma_\mu(H/\varrho\eta)) & \text{if } \varrho\eta H_{\varrho,\eta} \leq H < \varrho\eta H_\varrho, \\ \dfrac{\sigma_\mu(H_\varrho)}{\theta_\varrho(H_\varrho)} H & \text{if } \varrho\eta H_\varrho \leq H < \theta_\varrho(H_\varrho), \\ \sigma_\mu(\theta_\varrho^{-1}(H)) & \text{if } \theta_\varrho(H_\varrho) \leq H < \varrho\tau'_\mu(0), \\ \sigma_\mu(H/\varrho) & \text{if } \varrho\tau'_\mu(0) \leq H \leq \varrho\tau'_\mu(-\infty), \end{cases} \tag{2}$$

and $D_{\mu,\varrho,\eta}(H) = -\infty$ otherwise.

2. If $H_{\varrho,\eta} \geq H_\varrho$, then set

$$D_{\mu,\varrho,\eta}(H) = \begin{cases} \sigma_\mu(H/\varrho\eta) & \text{if } \varrho\eta\tau'_\mu(\infty) \leq H < \varrho\eta H_{\varrho,\eta}, \\ \sigma_\mu(\theta_\varrho^{-1}(H)) & \text{if } \varrho\eta H_{\varrho,\eta} \leq H < \varrho\tau'_\mu(0), \\ \sigma_\mu(H/\varrho) & \text{if } \varrho\tau'_\mu(0) \leq H \leq \varrho\tau'_\mu(-\infty), \end{cases} \tag{3}$$

and $D_{\mu,\varrho,\eta}(H) = -\infty$ otherwise.

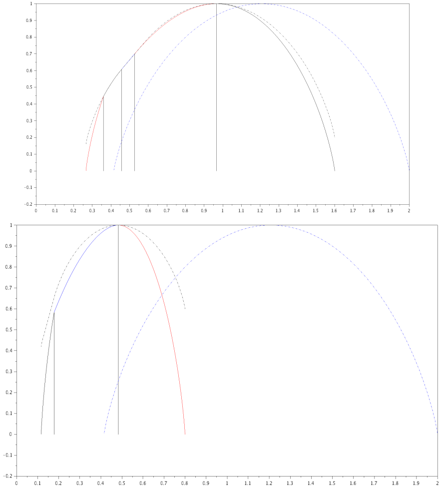

Figure: **Top:** Spectrum of $\mathsf{M}_{\varrho,\eta}$ for a given spectrum of μ when $\varrho = 0.8$, $\eta = 0.8$, case $\sigma_\mu(\tau'_\mu(\infty)) \leq \frac{1-\varrho}{1/\eta-\varrho}$ and $H_{\varrho,\eta} < H_\varrho$: 5 phases. **Bottom:** Spectrum when $\varrho = 0.7$, $\eta = 0.4$, case $\sigma_\mu(\tau'_\mu(\infty)) \leq \frac{1-\varrho}{1/\eta-\varrho}$ and $H_{\varrho,\eta} \geq H_\varrho$: 3 phases. Original spectrum in dashed blue, Legendre spectrum in dashed black.

Some explanations for the decreasing part of the spectrum

Recall that

$$\mathcal{S}_j = \{I \in \mathcal{D}_j : p_I = 1\}.$$

and for $x \in [0,1]$,

$$\delta_x = \limsup_{j \to \infty} \frac{\log d(x, \{x_I : I \in \mathcal{S}_j\})}{\log 2^{-j\eta}}.$$

Proposition

With probability 1, $\forall\, x \in [0,1]$, if $\delta_x = 1$, then $\underline{\dim}_{\rm loc}(\mathsf{M}_{\varrho,\eta}, x) = \varrho\, \underline{\dim}_{\rm loc}(\mu, x)$.

In particular, if $H = \varrho\tau'_\mu(q)$ for some $q \in \mathbb{R}$, then $\sigma_{\mathsf{M}_{\varrho,\eta}}(H) \geq \sigma_\mu(H/\varrho)$, since the unique ergodic Gibbs measure ν_q of maximal dimension supported on $E_\mu(\tau'_\mu(q))$ is such that $\nu_q(\{x : \delta_x = 1\}) = 1$.

Proposition

With probability 1, for all $H \in \mathbb{R}$, $\underline{\dim}_{\rm loc}(\mathsf{M}_{\varrho,\eta}, x)) = H$ implies that $\overline{\dim}_{\rm loc}(\mu, x)) \geq H/\varrho$.

This implies that $\sigma_{\mathsf{M}_{\varrho,\eta}}(H) \leq \sigma_\mu(H/\varrho)$ if $H \geq \varrho\tau'_\mu(0)$.
Consequently, $\sigma_{\mathsf{M}_{\varrho,\eta}}(H) = \sigma_\mu(H/\varrho)$ for all $H = \varrho\tau'_\mu(q)$, $q \leq 0$.

Some explanations for the increasing part of the spectrum

Definition (Conditioned ubiquity)

Let $h \geq 0$, $\delta \geq 1$, $\epsilon > 0$, and $\widetilde{\xi} := (\xi_j)_{j\geq 1}$ a non-decreasing positive sequence. Set

$$\mathcal{A}_{\varrho,h,\widetilde{\xi},\delta} = \bigcap_{J\geq 1} \bigcup_{j\geq J} \bigcup_{I\in\mathcal{S}_j:\, |I^{\varrho\eta}|^{h+\xi_j} \leq \mu(I^{\varrho\eta}) \leq |I^{\varrho\eta}|^{h-\xi_j}} B\Big(x_I, (2^{-j(\eta-\varepsilon_j)})^{\delta}\Big).$$

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Theorem

There exists a sequence $\widetilde{\xi}$ decreasing to zero such that for all $h > 0$ such that $\tau_\mu^*(h) > 0$ and all $\delta \geq 1$, there exists a Borel probability measure $\nu_{\varrho,h,\delta}$ such that $\nu_{\varrho,h,\delta}(\mathcal{A}_{\varrho,h,\widetilde{\xi},\delta}) = 1$ and

$$\underline{\dim}(\nu_{\varrho,h,\delta}) \geq \min\left(\frac{1-\varrho+\varrho\sigma_\mu(h)}{\delta}, \sigma_\mu(h)\right).$$

Also, $\dim \mathcal{A}_{\varrho,h,\widetilde{\xi},\delta} = \min\left(\frac{1-\varrho+\varrho\sigma_\mu(h)}{\delta}, \sigma_\mu(h)\right)$.

Some explanations for the increasing part of the spectrum

Recall that

$$D_{\varrho,\eta}(H) = \max\left\{ \min\left(\frac{1-\varrho+\varrho\sigma_\mu(h)}{\delta}, \sigma_\mu(h)\right) : 0 \le \frac{\varrho h}{\delta} \le H,\ 1 \le \delta \le 1/\eta \right\}.$$

For each $H \in [0, \varrho\tau_\mu'(0))$ such that $D_{\varrho,\eta}(H) > 0$, in fact $D_{\varrho,\eta}(H)$ is uniquely attained at some (h,δ) such that $\delta \in [1, 1/\eta]$, $\varrho h/\delta = H$. Also,

(1) $\mathcal{A}_{\varrho,h,\widetilde{\xi},\delta} \subset \underline{E}^{\le}_{\mathsf{M}_{\varrho,\eta}}(H) := \{x : \underline{\dim}_{\mathrm{loc}}(\mathsf{M}_{\varrho,\eta}, x) \le H\}$.

(2)

$$\underline{E}^{\le}_{\mathsf{M}_{\varrho,\eta}}(H) \subset \bigcap_{\varepsilon>0} \bigcup_{h\ge 0,\ \delta\in[1,1/\eta],\ \frac{\varrho h}{\delta}\le H+\varepsilon} \mathcal{A}_{\varrho,h,(\varepsilon)_{j\ge1},\delta},$$

which implies that

$$\sigma_{\mathsf{M}_{\varrho,\eta}}(H) \le \dim \underline{E}^{\le}_{\mathsf{M}_{\varrho,\eta}}(H) \le D_{\varrho,\eta}(H).$$

Since for all $k \ge 1$ one has $\dim \underline{E}^{\le}_{\mathsf{M}_{\varrho,\eta}}(H-1/k) \le D_{\varrho,\eta}(H-1/k) < D_{\varrho,\eta}(H)$, we get

$$\nu_{\varrho,h,\delta}(\mathcal{A}_{\varrho,h,\widetilde{\xi},\delta} \cap \underline{E}_{\mathsf{M}_{\varrho,\eta}}(H)) = 1.$$