Distribution of Stern–Brocot sequences generalized to Hecke triangle groups

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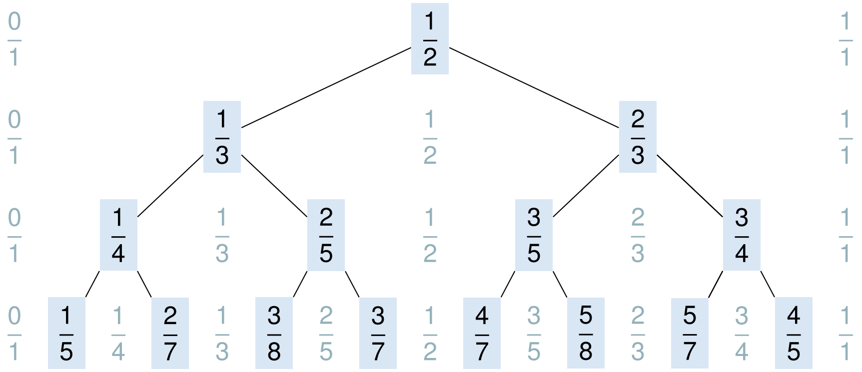
Thermodynamic Formalism: Non-additive Aspects and Related Topics

University of Bremen

Classical Stern–Brocot sequence as tree



- Calculate: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$
- $S_1 = \left\{\frac{1}{2}\right\}$, $S_2 = \left\{\frac{1}{3}, \frac{2}{3}\right\}, \ldots$
- Slow continued fraction algorithm (Richards 1981)
- All rational numbers

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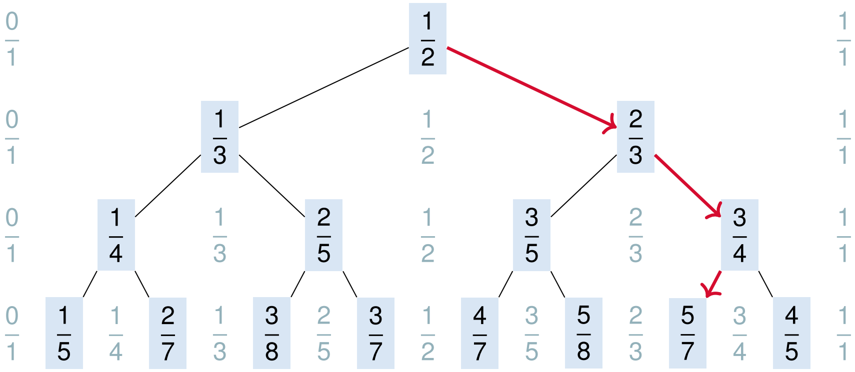

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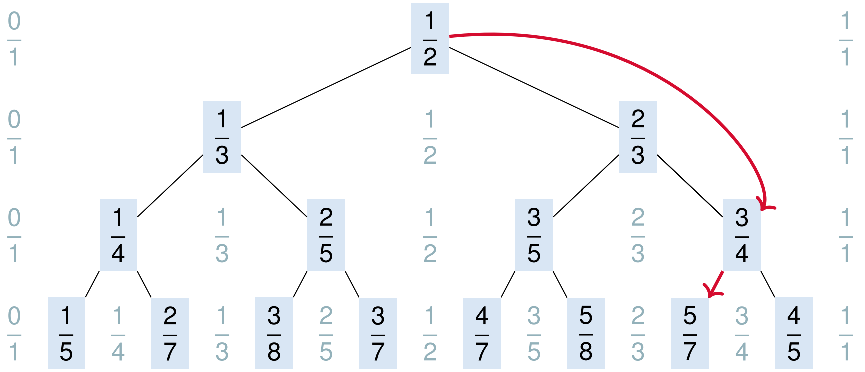

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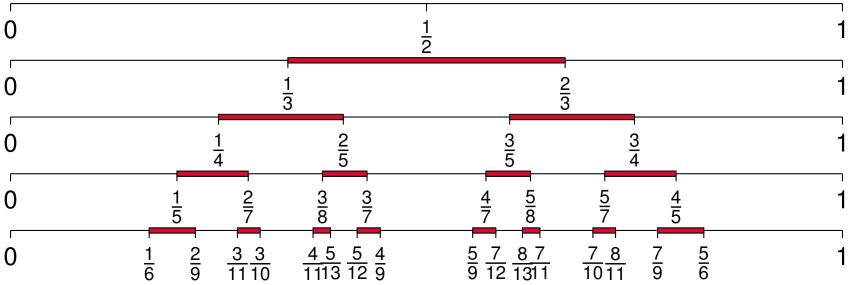
Distribution properties

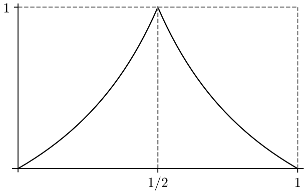


Points The Stern–Brocot sequence is not uniformly distributed (Keßeböhmer and Stratmann 2008) .

Sets The Lebesgue-measure m of the sequence of intervals vanishes as $\frac{\log(2)}{\log(n)}$ (Keßeböhmer and Stratmann 2012) .

The map describing the dynamics

- Recall: $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ via Möbius transformation.



The Farey map

$$F(x) := \begin{cases} \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right].x, & x \in \left[0, \tfrac{1}{2}\right], \\ \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right].x, & x \in \left[\tfrac{1}{2}, 1\right], \end{cases}$$

satisfies $F^{-(n-1)}\left(\left\{\frac{1}{2}\right\}\right) = S_n$.

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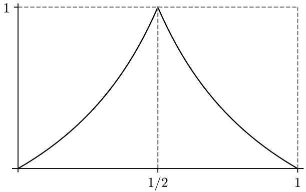

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satisfies $F^{-(n-1)}\left(\left\{\frac{1}{2}\right\}\right) = S_n$.

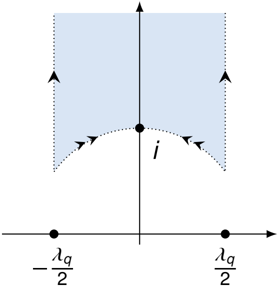
- Keßeböhmer and Stratmann (2012): For $[\alpha, \beta] \subset (0, 1]$,

$$m(F^{-n}([\alpha, \beta])) \sim \frac{\log\left(\frac{\beta}{\alpha}\right)}{\log(n)}.$$

  - Further dynamical results, including the distribution of the Stern–Brocot sequence

Motivation

- In Keßeböhmer and Stratmann (2012): $\mathrm{PSL}(2, \mathbb{Z})$-action.
- Does arithmeticity of $\mathrm{PSL}(2, \mathbb{Z})$ determine the behavior of F?



- Idea: Consider groups with similar geometric structure, but not arithmetic
- Based on these groups we define generalized Farey maps T_q.

Main result: Dynamics of generalized Farey maps

- m Lebesgue-measure on $[0,1]$
- T_q generalized Farey map

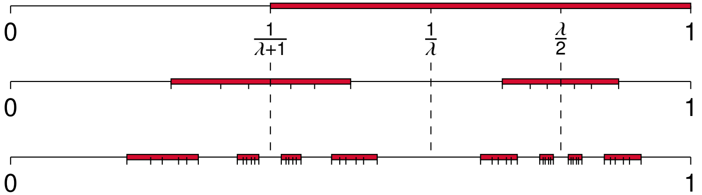
Theorem (B.–Keßeböhmer–Pohl)

Let $q \geq 3$ be odd. For every $[\alpha,\beta] \subset (0,1]$ we have

$$\underset{n\to\infty}{*\text{-}\lim}\left(\frac{\log(n)}{\log\left(\frac{\beta}{\alpha}\right)} \cdot m|_{T_q^{-n}([\alpha,\beta])}\right) = m\,.$$

Particularly, $m(T_q^{-n}([\alpha,\beta])) \sim \frac{\log\left(\frac{\beta}{\alpha}\right)}{\log(n)}$ for $n \to \infty$.

$q = 5$:

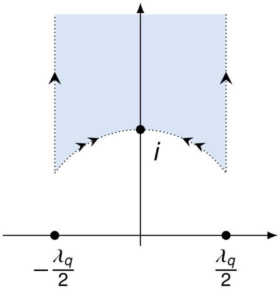


How to generalize the Farey map?

Definition

Let $q \in \mathbb{N}$, $q \geq 3$ and $\lambda_q := 2\cos\left(\frac{\pi}{q}\right)$. The Hecke triangle group Γ_q is generated by

$$S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad U := \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix}.$$



A fundamental domain for Γ_q

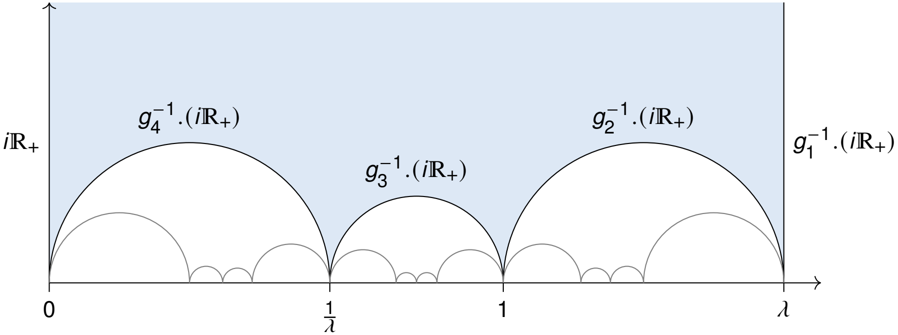
- Discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$
- $\Gamma_3 = \mathrm{PSL}(2, \mathbb{Z})$

Geometric motivation

- Discretization of geodesic flow on hyperbolic surface by Pohl (2016); yields

$$g_k := ((US)^k S)^{-1}$$

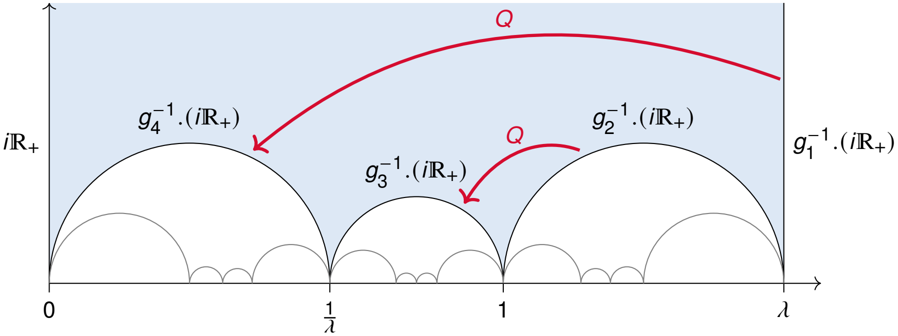
where $S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $U := \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix}$.



Tessellation for $q = 5$

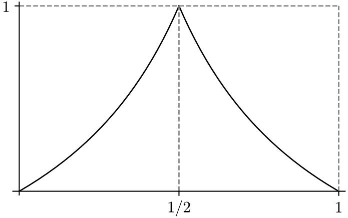
Geometric motivation

- The generalized Farey map should be a transformation on $[0, 1]$.
- Symmetry given by $Q.z = 1/\overline{z}$

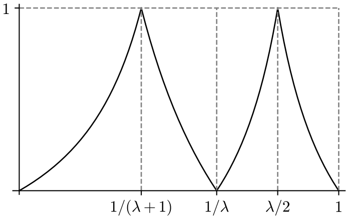


Tessellation for $q = 5$

Farey map generalized to Hecke triangle groups



$q = 3$



$q = 5$

With $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$$F(x) = \begin{cases} g_2.x, & x \in \left[0, \frac{1}{2}\right], \\ Qg_2.x, & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

For odd $q \geq 3$,

$$T_q(x) := \begin{cases} g_{q-1}.x & x \in g_{q-1}^{-1}.[0,1], \\ Qg_{q-1}.x & x \in g_{q-1}^{-1}Q.[0,1], \\ & \vdots \\ g_{\frac{q+1}{2}}.x & x \in g_{\frac{q+1}{2}}^{-1}.[0,1], \\ Qg_{\frac{q+1}{2}}.x & x \in g_{\frac{q+1}{2}}^{-1}Q.[0,1]. \end{cases}$$

We have $F = T_3$.

Generalized Stern–Brocot sequence

- Classical Stern–Brocot sequence:

$$S_n = F^{-(n-1)}\left(\left\{\tfrac{1}{2}\right\}\right)$$

- Note: $\left\{\frac{1}{2}\right\} = F^{-1}(\{0,1\}) \setminus \{0,1\}$
- Use the generalized Farey map T_q.

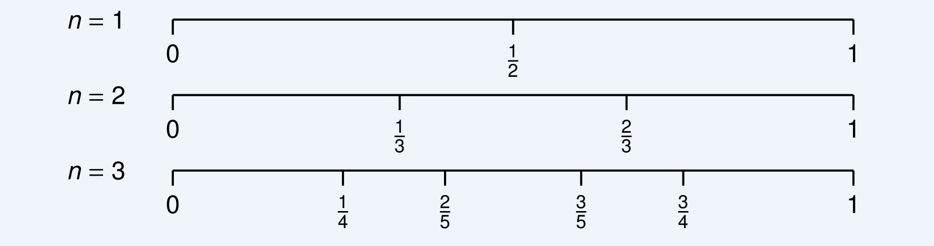
Definition

The generalized Stern–Brocot sequence $(S_n)_{n\in\mathbb{N}}$ for odd $q \geq 3$: $S_{-1} := \emptyset$,

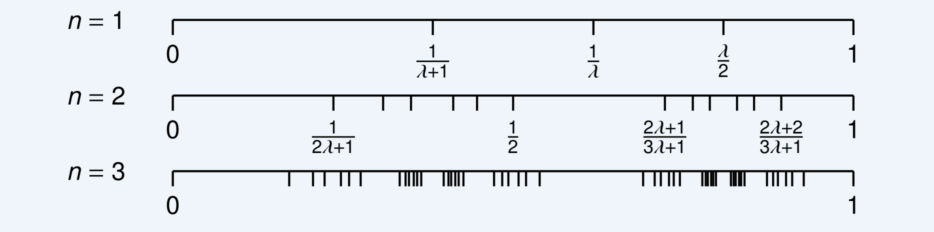
$$S_n := T_q^{-n}(\{0,1\}) \setminus S_{n-1}$$

Examples

q=3



q=5



A property of the Stern–Brocot elements

Proposition (B.)

The union of all Stern–Brocot elements is $\Gamma_q.\infty \cap [0,1]$.

- Matrix coefficients of Γ_q are elements of $\mathbb{Z}[\lambda_q]$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.\infty = \frac{a}{c}\,.$$

q=3: We have $\Gamma_3.\infty \setminus \{\infty\} = \mathbb{Q}$.

q=5: We have $\Gamma_5.\infty \setminus \{\infty\} = \mathbb{Q}(\sqrt{5})$ (Leutbecher 1967).

- $\lambda_5 = \frac{1}{2}\left(1 + \sqrt{5}\right)$

Result: Distribution of Stern–Brocot sequences

- Let W_n be the words of length n to the alphabet of inverse branches of T_q,

$$\left\{ g_k^{-1}, g_k^{-1} Q : k = \tfrac{q+1}{2}, \ldots, q-1 \right\} .$$

Theorem (B.–Keßeböhmer–Pohl)

Let $q \geq 3$ be odd. For each $x \in (0,1)$ we have

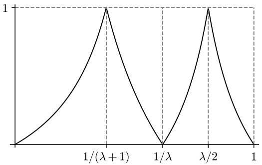
$$\underset{n\to\infty}{*\text{-}\lim}\, x \log(n) \sum_{g \in W_n} |g'(x)| \, \delta_{g.x} = m .$$

- A sequence $(x_i)_{i\in\mathbb{N}}$ in $[0,1]$ is uniformly distributed w.r.t. m if

$$\underset{n\to\infty}{*\text{-}\lim}\, \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} = m .$$

Behavior of generalized Farey map

The generalized Farey map T_q has the properties:



- Piecewise monotonic
- Adler's condition: The map $(T_q)''/(T_q')^2$ is bounded
- Non-uniformly expanding:
  - $T_q(0) = 0$
  - $T_q'(0) = 1$
  - $|T_q'| \geq \rho(\varepsilon) > 1$ on $[\varepsilon, 1]$
- Topologically mixing

Result

The generalized Farey map T_q is ergodic and conservative with respect to the Lebesgue measure.

Central: Transfer operator

Definition

Let $(X, \mathcal{B}, \eta, T)$ be a dynamical system. The transfer operator $\widehat{T}_\eta : L^1(\eta) \to L^1(\eta)$ is the operator that satisfies for all $B \in \mathcal{B}$, $f \in L^1(\eta)$

$$\int_B \widehat{T}_\eta(f)\, d\eta = \int_{T^{-1}(B)} f\, d\eta\,.$$

- Strategy: Find a measure $d\mu = h\, dm$ such that uniformly on all $[\alpha, \beta] \subset (0,1]$, for $g = f/h$ with f Riemann integrable,

$$\lim_{n\to\infty} \log(n) \left(\widehat{T}_q\right)^n_\mu (g) = \int_{[0,1]} g\, d\mu\,.$$

- With some prerequisites, this is a consequence of a result from Melbourne and Terhesiu (2012) .

Why? The proof then goes like this...

- $d\mu = h\, dm$
- For all $[\alpha, \beta] \subset (0, 1]$, $f \in C([0, 1])$,

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{T_q^{-n}([\alpha,\beta])} \frac{\log(n)}{\log(\beta/\alpha)} f\, dm \\
&= \lim_{n\to\infty} \int_{T_q^{-n}([\alpha,\beta])} \frac{\log(n)}{\log(\beta/\alpha)} f/h\, d\mu \\
&= \lim_{n\to\infty} \int_{[\alpha,\beta]} \frac{\log(n)}{\log(\beta/\alpha)} (\widehat{T}_q)_\mu^n (f/h)\, d\mu \\
&= \left(\int_{[0,1]} f/h\, d\mu \right) \cdot \underbrace{\int_{[\alpha,\beta]} \frac{1}{\log(\beta/\alpha)}\, d\mu}_{=1 \text{ (by choice of } h)} .
\end{aligned}
$$

How to obtain a suitable density *h*?

Proposition (B.–Keßeböhmer–Pohl)

Let

$$h(x) := \frac{1}{x}.$$

For all $q \geq 3$ we have $\left(\widehat{T}_q\right)_m h = h$.

- Let $d\mu = h\,dm$, then the transfer operators are conjugate to each other,

$$\left(\widehat{T}_q\right)_\mu (f) = \frac{1}{h}\left(\widehat{T}_q\right)_m (f \cdot h).$$

- μ is T_q-invariant if

$$\mathbb{1} = \left(\widehat{T}_q\right)_\mu (\mathbb{1})$$

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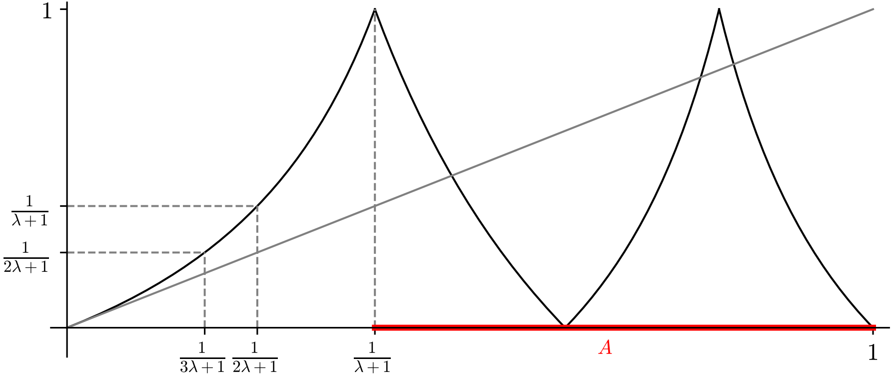
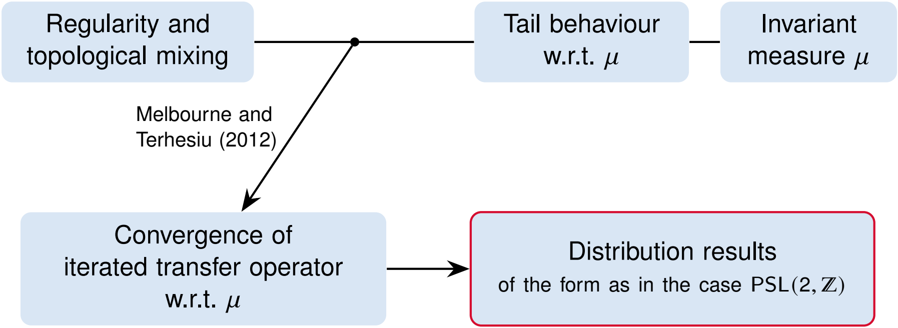
Tail behavior of the measure μ

- Return time: $\varphi(x) = \min\{n \geq 1 : T_q^n(x) \in A\}$

$$\mu(\{\varphi > n\} \cap A) = \mu\left(\{\varphi = n\} \cap A^C\right) = \mu\left(\left[\frac{1}{(n+1)\lambda+1}, \frac{1}{n\lambda+1}\right]\right) \sim \frac{1}{n}$$



Wrapping up



Takeaway: Not the arithmeticity decides the distribution behavior, but the dynamics and geometry of the underlying group.